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Modelling, regularisation, and analysis of Dean–Kawasaki type equations

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Declaration of any previous Submission of the Work

The material presented here for examination for the award of a higher degree by research has not been incorporated into a submission for another degree.

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Federico Cornalba

Declaration of Authorship

I am the author of this thesis, and the work described therein was carried out by myself personally, with the exception of Chapters 2 to 4, which contain research articles that originated from collaboration with my supervisors Johannes Zimmer and Tony Shardlow.

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Federico Cornalba

Summary

We investigate modelling and analysis for a specific class of stochastic equations arising in fluctuating hydrodynamics: this class, which we refer to as the *Dean–Kawasaki* (DK) class, is broadly concerned with the description of mesoscopic fluctuations in finite-size particle systems.

We focus on two notable members of this class. The first one, to which most of the thesis is devoted, is the DK equation. We revisit its original derivation from physics in a mathematically rigorous way, by considering particles of finite rather than atomic size. We do this in the two relevant cases of independent particles and of particles weakly interacting via a pairwise potential. In both cases, we derive a regularised DK model in the form of a stochastically perturbed wave equation. For this model we establish high-probability existence and uniqueness results by using small-noise techniques.

The issue of almost-sure positivity of solutions (a critical feature for the DK class) motivates the final part of the thesis: there, we study a second member of the class, namely, a stochastic thin-film equation. We provide sufficient conditions on the interplay of stochastic noise and the source potentials in order to extend a positive local solution (defined up to a stopping time) up to any deterministic time, and we draw relevant analogies with the existing literature and with the DK equation.

Finally, we detail possible directions for future work.

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Contents

Chapter 1

Introduction

We study a class of stochastic partial differential equations, which we refer to as the *Dean–Kawasaki class* (DK class for short). This class, whose distinctive features will be pointed out in due course and whose very name will be justified shortly, plays an important role in the theory of *fluctuating hydrodynamics* [22]. This theory is concerned with the mathematical description of the evolution of specific systems made of a finite number of particles: the defining and fundamental feature of the systems in question is the exhibition of intrinsic random fluctuations at the particle level.

In simple terms, one can think of any of these systems as being a collection of elements (such as particles, individuals, animals, etc.), whose evolution in time is primarily influenced by three distinct factors: (i) influence coming from outside the collection itself, usually consisting of suitable external fields; (ii) influence coming from within the collection, typically consisting of some kind of interaction between different elements; (iii) random fluctuations affecting the collection's elements (e.g., thermal fluctuations). Feature (iii) is crucial, as anticipated above. See Figure 1-1, *left image*.

Features (i)–(ii) reflect the *deterministic* component of the system evolution, i.e., the dynamics (uniquely determined by the past and current states of the system) which would naturally occur should the *stochastic* component (feature (iii)) be absent. The stochastic component results in deviations from the deterministic dynamics, and can lead to non-trivial phenomena: as a notable example, we mention the metastable transition times for stochastic systems with multistable potentials, which are in many cases studied using Kramers' law [39, 25, 5].

The contents of this work broadly revolve around the analysis of the stochastic component of certain particle systems, to be specified below.

Examples of particle systems whose general characteristics are as described above are numerous, and can be found in many important fields. These fields include the theory of *Newtonian fluids* (molecules in a thin-liquid film [49, 42]), of *active matter* [9] (real life groups, such as fish schools, bird flocks, bacterial colonies [58]), of *thermal advection* (particles interacting with a heat bath/solvent [55, 44, 41], diffusive passive tracer particles [17]), of *neural networks* (auxiliary particle systems associated with the analysis of the loss function landscape of the network [54]).

Any given particle system abides by elementary laws (such as, for instance, *classical mechanics* laws), which prescribe the motion of the individual particles. While this is an accurate representation of the system, it is also a computationally inefficient one. This is why one normally chooses to formulate *evolution equations* which can effectively describe the particle systems on more coarse-grained *length scales*, by keeping track of fewer meaningful quantities. The length scale of the DK class, whose members are indeed evolution equations, is suitable for capturing the systems' ensemble fluctuations. More details are given in Section 1.1.

In this thesis we study two incarnations of the DK class. These incarnations are closely related to some evolution equations which have been proposed in the last two decades with respect to the description of two particle systems of relevance. These systems are: a *Langevin* (LA) particle system; a system of molecules in a *thin-film* (TF) liquid. As for the LA system, an evolution equation was proposed by D. Dean and K. Kawasaki in the late 90ies [15, 32]: it is accordingly referred to as *Dean–Kawasaki*

(DK) equation (or model), and reads

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \nabla W * \rho) + \Delta \rho + \nabla \cdot (\sigma \sqrt{\rho}\, \xi)\,, \tag{1.1}$$

for particle density ρ, particle interaction potential W, and stochastic driving force ξ. The DK equation, which we will later on take as 'reference model' for our definition of the DK class (Section 1.5), shares many crucial similarities with the evolution equation for the TF system, which is called the stochastic *thin-film* (TF) equation (thus also in the DK class), and reads

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \left(\rho^n \nabla \left\{\Delta \rho - W'(\rho)\right\}\right) + \nabla \cdot \left(\sigma \sqrt{\rho^n}\, \xi\right), \tag{1.2}$$

for some $n \in \mathbb{N}$ and some interface potential W. We have introduced these two equations at this early stage for the sake of context, and precise details will be given throughout this introductory chapter.

Despite a long-standing interest of the physics community in these two equations, rigorous mathematical results are few. A non-exhaustive list of the researchers who have contributed towards the mathematical understanding of these equations includes M. von Renesse and collaborators for the DK equation, and J. Fischer, G. Grün and B. Gess for the TF equation. Specific details are given in Section 1.6.

The main contributions of this thesis are as follows. We derive a *regularised* incarnation of the DK equation based on the LA system. We do this both in the case of *non-interacting* particles and particles *weakly interacting* via a *pairwise* potential. This regularised model, which we then analyse, addresses some regularity issues of the original DK model. Furthermore, a relevant open question shared by the original and regularised DK models (i.e., the issue of positivity of solutions) is then framed in the wider context of modifications of the TF equation. These modifications, which are within the DK class, allow us to discuss the positivity issue more effectively, and in greater generality. An *a priori* analysis of positivity of solutions for these modified TF equations is performed, giving us useful insight on the DK class.

1.1. Relevant length scales

One may define the evolution equation (describing a particle system) in one of the three major length scales: *microscopic*, *mesoscopic*, or *macroscopic*, see Figure 1-1. The microscopic scale refers to the level of individual elements of the system: this means that the dynamics of each single particle is kept track of in the model. In particular, each particle is distinguishable from any other. This constitutes the finest, most accurate, but often most computationally burdensome level to which the system can be studied. On a coarser scale, we find the mesoscopic level, in which summarising quantities (such as densities, averages, etc.) are introduced in the model, and in which we also retain some degree of information coming from the microscopic scale. Within the mesoscopic scale, singling out specific elements (and their fluctuations) is not possible, while the ensemble fluctuations are observable on top of the 'average' deterministic evolution. Finally, by further zooming out (usually, by performing a suitable hydrodynamic limit $N \to \infty$ [33]), one finds the macroscopic scale, in which nothing but summarising, global coarse-grained quantities are used to describe to system: typically, on this scale, the random fluctuations of the particles are neglected, and the resulting equation is less accurate but substantially simpler to analyse.

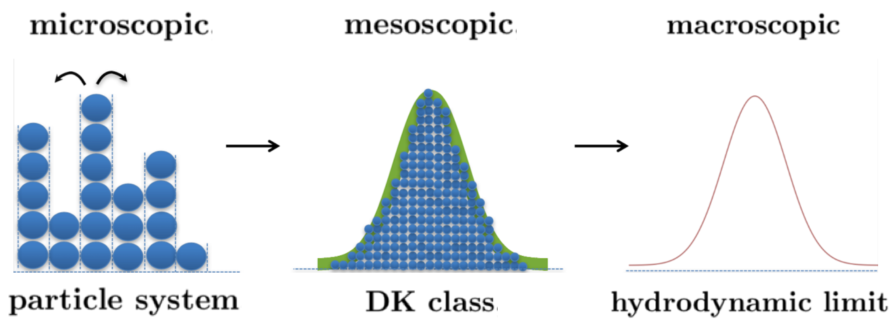

Figure 1-1: ***Left****: An example of a particle system, where the effect of the random fluctuations can be seen in the particles at the top of the piles being able to randomly move either to the right-adjacent or the left-adjacent pile.* ***Centre****: Empirical density for the same particle system, for a large number of particles N. The ensemble fluctuations are visible on top of the 'average' profile.* ***Right****: The stochastic fluctuations are neglected in the hydrodynamic limit $N \to \infty$, giving only the 'average' density of particles.*

The DK class, as previously hinted, is intrinsically set in the mesoscopic scale, as its defining goal is to reflect the particle fluctuations.

1.2. Particle systems of interest

We now briefly describe the LA and TF particle systems. It is worth pointing out that we will be working with the exact mathematical microscopic description of the LA system (in Chapters 2 and 3) in order to provide our results on a regularised DK model (i.e., on the mesoscopic scale). On the other hand, our results on suitable modifications of the stochastic thin-film equation (Chapter 4) are motivated by analytical analogies with the DK model. Consequently, we provide accurate details on the microscopic level for the LA system, while we limit ourselves to giving some general, concise, context for the TF system.

1.2.1 Langevin (LA) particle system

Consider a collection on N particles moving in $\mathbb{R}^d$, where $d \in \mathbb{N}$. Any given particle is subject to a frictional drag caused by its motion in the surrounding environment, to energy fields (giving interactions with the environment, or with other particles), and to random fluctuations. Systems of this type are conceptually very simple and general, and are thus found in many fields [55, 44, 17]: they are generically referred to as *Langevin* particle systems. In this thesis, we will be working with one-dimensional models ($d = 1$), with particles identified by positions and velocities $(\mathbf{q}, \mathbf{p}) = (q_i, p_i)_{i=1}^N$, and where the motion of a single particle $i = 1, \ldots, N$ is given by the stochastic equation (SDE)

$$\dot{p}_i = q_i, \quad \dot{p}_i = -\gamma p_i - \nabla \left[\mathbf{W}(\mathbf{q}, \mathbf{p})\right]_i + \sigma \beta_i, \tag{1.3}$$

for (stochastically) independent initial conditions, independent Gaussian driving forces $\{\beta_i\}_{i=1}^N$, frictional constant $\gamma > 0$, noise amplitude σ, and energy field $\mathbf{W}$. The first equation in (1.3) is simply the definition of velocity, while the second one reflects Newton's second law of motion. We are interested both in the *independent* particles case (associated with an *on-site* potential $\left[\mathbf{W}(\mathbf{q}, \mathbf{p})\right]_i = V(q_i)$, for some suitable potential V)

and in the *weakly interacting* particles case (given by a nonlocal interaction potential $[\mathbf{W}(\mathbf{q},\mathbf{p})]_i = N^{-1}\sum_{j=1}^{N} V(q_i - q_j)$).

1.2.2 Thin-film (TF) particle system

A thin-liquid film corresponds to a liquid layer with a small number of particles in thickness (usually no more than $10^2 - 10^3$), and sitting on top of a (fixed) substrate layer. The thin-film surface is a free surface. Relevant phenomena that can be observed include *droplet spreading* (the process through which the thin-film diffuses over the substrate) and *dewetting* (the process through which a thin-liquid film gradually retracts from the substrate).

The main microscopic features characterising this type of particle system are: (a) interaction between thin-liquid film and substrate molecules; (b) thin-liquid film intermolecular interaction; (c) capillarity effects; (d) surface tension effects on the free surface; (e) thin-film molecules thermal fluctuations (of Gaussian type); (f) local source potentials.

Such microscopic dynamics is more complex than that of the LA system, and experimentally much harder to simulate [2]. Therefore, a system description based on continuum mechanics [1, 49, 43] is almost always preferred in the case of thin-liquid film. Because of this, we do not provide any explicit microscopic mathematical description for this system.

1.3. Physical features of LA/TF systems

The LA and TF systems share the following distinctive physical properties.

1.3.1 Mass preserving fluctuations

The LA and TF systems are subject to fluctuations which, on their own, do not alter the total mass of the system (i.e., the total number or individuals). In the case of the LA system, the total mass is also conserved (as the number of particles N is kept fixed), whereas, for the TF system, the mass can be inserted/removed from the system through deterministic local source potentials.

1.3.2 Fluctuation-dissipation relation

The LA and TF systems satisfy a *fluctuation-dissipation* relation. In other words, there is a suitable balance between the damping deterministic dynamics of the particle system and the magnitude of the particles' random fluctuations. This balance results in the particles' dynamics converging, in a characteristic relaxation time, to a steady fluctuating configuration in equilibrium [40]. As for the LA system, for which we have provided a mathematical microscopic description, the fluctuation-dissipation relation arises from the suitable balance of friction $-\gamma p_i$ and noise $\sigma\dot{\beta}_i$, as from the Ornstein-Uhlenbeck setting [18]: more precisely, the characteristic ratio $\beta := 2\gamma/\sigma^2$ (known as *inverse temperature* of the system) gives shape to a steady configuration $f(q,p) \propto \exp\{-\beta S(q,p)\}$, where S is the sum of the kinetic and potential energy of the system.

1.4. Mathematical implications on the DK class

The physical features of the LA and TF systems are reflected in specific mathematical features for the associated evolution equations, which belong to the DK class. These features are sufficient to provide a general description of the stochastic component found in the DK class. For the sake of exposition, we colloquially describe these features prior to giving their precise mathematical declination within the DK class.

1.4.1 Conservative stochastic component

In order to reflect the mass-preserving fluctuations, the stochastic component of the DK class is framed within a spatial divergence structure, and boundary conditions are understood to be periodic. The action of the divergence operator is often only formal, and we will be more specific later on.

1.4.2 Infinite-dimensional noise representation

As pointed out earlier, the DK class is concerned with a mesoscopic representation of an underlying particle system. As the particles can not be individually traced on this scale (see Section 1.1), we need to characterise the ensemble fluctuations via a single, general fluctuating term. Since the particles move in time and space, we need such fluctuating term to act *both* in time in space, hence the infinite-dimensionality requirement. Depending on the chosen spatial correlation that one wants to prescribe for the Gaussian forces driving the particles, one typically relies on a suitable infinite-dimensional Gaussian noise, as clarified in due course.

1.4.3 Noise dependency on particle density

The Gaussian noise introduced in Subsection 1.4.2 is the random driving force of the DK class, and is therefore included in the aforementioned spatial divergence form of the stochastic component of the DK class. The remaining ingredient for the stochastic component is the so-called *mobility coefficient*. This coefficient, whose arguments are suitable macroscopic quantities describing the particle system, acts *multiplicatively* on the deterministic and the stochastic component of the DK class. The mobility coefficient 'tunes' the effectiveness of both components, in order to reflect relevant physical properties of the particle system. These properties include: (i) the previously mentioned fluctuation-dissipation relation; (ii) density-dependent phenomena, such as obvious absence of fluctuations in spatial regions not containing particles, or the noise amplitude being monotonically increasing with the particle density.

1.5. Dean–Kawasaki class

We give mathematical substance to the DK class. Being a mesoscopic representation of underlying particle systems, the DK class describes the effect of particle fluctuations on the systems' macroscopic quantities, the most important of which is the *particle density* $\rho = \rho(x,t)$, where $(x,t) \in \mathbb{R}^d \times [0,T]$. On a *formal* level, a generic member of the DK class reads

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left(m(\rho) \nabla \frac{\delta F[\rho]}{\delta \rho} \right) + \Gamma[\rho] + \nabla \cdot \left(\sigma \sqrt{m(\rho)} \xi \right) =: \mathscr{D}_1 + \mathscr{D}_2 + \mathscr{S}. \tag{1.4}$$

for some positive function m, functionals F and Γ, and stochastic noise ξ, all to be discussed in the next subsection.

1.5.1 General features

We explain how the structure of the DK class (1.4) reflects the previously mentioned physical and mathematical properties of the LA and TF systems, given in Sections 1.3 and 1.4.

Equation (1.4) comprises a deterministic component (or *drift*) $\mathscr{D} := \mathscr{D}_1 + \mathscr{D}_2$ and a stochastic component $\mathscr{S}$. The diffusive (mass-preserving) term $\mathscr{D}_1$ accounts for the gradient-flow dynamics of the particle system according to the steepest descent of an energy functional F [31]. The functional F encodes the dynamics resulting from local and nonlocal particle interactions, but does not account for mass introduction/removal in the system. As a notable example, choosing F as the Gibbs–Boltzmann entropy functional [31] gives the standard diffusive term $\mathscr{D}_1 := \Delta\rho$. The diffusive term $\mathscr{D}_1$ is multiplicative in the *mobility coefficient* $m(\rho)$ (e.g., see [8]), which we have introduced in Section 1.4. This coefficient is responsible for diversifying the response of the equation with respect to different density profiles, corresponding to different macroscopic phenomena for the particle system. Relevant profiles are usually those of low density (i.e., $\rho \approx 0$) and high density (i.e., ρ close to a saturation value, either finite or infinite).

The non-conservative term $\mathscr{D}_2 \equiv \Gamma$ is a local source which can introduce/remove mass from the system.

Finally, the term $\mathscr{S}$ is the stochastic component, accounting for a mesoscopic representation of the particle fluctuations. The features pointed out in Section 1.4 are quite evident. Firstly, the noise is in divergence form, representing conservation of mass (provided suitable boundary conditions are also prescribed). Secondly, the fluctuations are given under a unifying, driving force ξ, which is an infinite-dimensional Gaussian noise. For example, ξ might be a *cylindrical* Wiener process (also called *space-time white noise*), or a Q-Wiener process [52]. More technical discussions on the two different types of noise are deferred to Chapters 2, 3 and 4. Thirdly, the noise is multiplicative in $(m(\rho))^{1/2}$, and this represents two things: the fluctuation-dissipation nature of the particle system (to be seen in the m-exponents being 1 and 1/2 in $\mathscr{D}_1$ and $\mathscr{D}_2$, respectively [22]); different noise amplitude for different density profiles (similar discussion for $\mathscr{D}_1$).

1.5.2 Original DK equation and TF equation

As mentioned earlier, we are interested in two notable members of the DK class, namely (1.1) and (1.2). Equation (1.1), which is the original *Dean–Kawasaki (DK)* equation [15, 32], corresponds, in the notation introduced in (1.4), to *linear* mobility ($m(\rho) = \rho$), null source potential ($\Gamma = 0$), space-time white noise ξ, and energy

$$F[\rho] = F_1[\rho] + F_2[\rho] := \int_{\mathbb{R}^d} (W * \rho)(x)\rho(x)\mathrm{d}x + \int_{\mathbb{R}^d} \rho(x)\log(x)\mathrm{d}x$$

reflecting particle interaction (via the potential W in F_1) and particle diffusion (through the Gibbs–Boltzmann entropy functional F_2 [31]).

Equation (1.2) is the stochastic *thin-film* equation, corresponding to a monomial mobility $m^n(\rho)$ (typically cubic [28, 14]), null source potentials ($\Gamma = 0$), Gaussian noise ξ coloured in space and white in time, Ginzburg-Landau type [8] energy functional

$$F[\rho] := \int_{\mathbb{R}^d} \{|\nabla\rho(x)|^2 + W(\rho(x))\}\mathrm{d}x$$

reflecting the effective interface potential between thin-liquid film and substrate (through

W) and balance of forces on the thin-liquid film free surface (through $|\nabla\rho|^2$).

1.5.3 General mathematical criticalities

Members of the DK class, including the DK equation (1.1) and the stochastic TF equation (1.2), are widely simulated in physics, see for example [58, 20, 44]. On the other hand, few rigorous mathematical results are available in this class. According to the specific instance (i.e., for a given choice of mobility, energy functional, etc....), one can find different factors which contribute to making the analysis challenging. However, most of these factors are pretty much built in the class, and have to be dealt with regardless of the specific instance. We describe them briefly.

The first issue is the divergence form of the stochastic noise. While this makes physical sense, it is a mathematical inconvenience. In particular, one needs to give a rigorous meaning to the action of the divergence operator on the infinite-dimensional multiplicative noise $\sqrt{m(\rho)}\xi$. In addition, this differential feature neatly distinguishes the analysis of the DK class from that of most equations with multiplicative noise. The most notable example of such equations is associated with the Super-Brownian motion (or Dawson–Watanabe process [60]), and it reads [38, 53]

$$\frac{\partial\rho}{\partial t} = \Delta\rho + \sqrt{\rho}\,\xi, \tag{1.5}$$

where ξ is a space-time white noise. The existence analysis of (1.8) relies on a variational formulation of a suitable martingale problem, which crucially benefits from the regularity of the Green function of the Laplacian operator [56] acting on the noise (in an L^2-duality sense). The case of (1.4) is radically different, as the additional spatial derivative in the noise diminishes the regularisation coming from the Green function.

The second issue is given by the natural positivity requirement on the density ρ. With very few exceptions, which will be mentioned later, there is yet no evidence that the DK class preserves positivity of solutions.

A third issue is the mobility itself. Typically being a monomial, m vanishes for null density (i.e., is degenerate). Thus, in a low-density regime, it annihilates the action of both deterministic drift and stochastic component. These facts, together with the positivity constraint on ρ, typically impose the definition of challenging solution function spaces, where basic properties (such as linear structure, convexity) might get lost. In addition, with the exception of quadratic mobility, all polynomial mobilities either give a non-Lipschitz noise ($m(\rho) = \rho^\gamma$, $\gamma < 2$, as for (1.1)) or superlinear noise ($m(\rho) = \rho^\gamma$, $\gamma > 2$, as for relevant instances of (1.2)).

Finally, the Gaussian noise ξ can be irregular (either with large or infinite trace [52]).

1.6. Existing literature for DK and stochastic TF equations

As pointed out earlier, the DK equation does not, by any means, lend itself to a straightforward mathematical analysis. Due to the low regularity of the noise, most definitions of solutions for stochastic partial differential equations (SPDEs) are not suitable to work with. In particular, the combination of non-Lipschitz and divergence form of the stochastic component, and the space-time white noise seemingly prevent from looking for strong solutions [52], mild solutions [13], or solutions in the context of paracontrolled distributions theory [29] and rough paths theory [45, 26].

As of today, the most widely accepted analytical setting sees the DK equation (1.1) as a stochastic perturbation of a gradient-flow dynamics with respect to a Wasserstein space, with the noise being aligned with Otto's formal Riemannian structure for optimal transportation [50]. In a nutshell, on a formal level, the short dynamics of the equation is governed by a *large deviation* principle with rate

$$\mathcal{A}(\rho) = \int_0^1 \sup_{\phi} \left\{ \langle \partial_t \rho, \phi \rangle - \mathcal{H}(\rho, \phi) \right\} \mathrm{d}t. \tag{1.6}$$

In the above, ϕ spans a suitable test function space, the brackets $\langle , \rangle$ refer to measure/function L^2-duality, and $\mathcal{H}$ is a rescaled *Hamiltonian* (reflecting the short time evolution of the equation). The rate functional $\mathcal{A}$ relates the likelihood of observing a given trajectory ρ to the amplitude of the stochastic noise of the equation. This rate is a crucial tool in the understanding of the equation's dynamics. We do not provide any additional details on rate functionals and Large Deviation theory [21]. In the specific case of the DK equation, the Hamiltonian is $\mathcal{H}(\rho, \phi) = \langle \rho, |\nabla \phi|^2 \rangle$ (which can also be seen as the *quadratic variation* of the noise of (1.1)); the link with the Wasserstein geometry is thus given by the use of the *Benamou-Brenier* formula on $\mathcal{H}$ [3].

In this general setup, the natural definition of solution is a *measure-valued martingale* solution. The key advantage, in this setting, is that the quadratic variation $\mathcal{H}$ removes the square-root singularity of the DK noise in (1.1) (simply by squaring it), thus giving a simpler object to analyse as a measure/test function duality.

Several results have been produced, mostly by von Renesse and coworkers. Firstly, a process having the Wasserstein distance as core evolution metric was found for the following non-conservative perturbation of the purely diffusive DK equation [59]

$$\frac{\partial \rho}{\partial t} = \alpha \Delta \rho + \Gamma[\rho] + \nabla \cdot \left(\sigma \sqrt{\rho}\, \xi \right), \tag{1.7}$$

where Γ is a specific nonlinear operator. A wider range of possibilities for Γ, all leading to Wasserstein-short-time dynamics, were later proposed in [37, 36, 48].

The need for a non-conservative correction Γ in (1.7) has been recently understood [35]. More precisely, if one does not allow for such a correction, then (1.7) either admits a unique atomic solution or no solution at all, depending on the diffusion scale $\alpha > 0$. An analogous result was later proved for (1.7) enriched with a mean interaction field [34]. As a result, no smooth solutions exist for the original DK equation (1.1). The results in [34, 35] are based on a special interplay between the quadratic variation $\mathcal{H}(\rho, \phi)$ and the Laplacian operator. Such interplay is provided by a straightforward application of the Itô formula on a variational formulation of (1.1).

With the results [34, 35] in mind, one can appropriately *a posteriori* frame the literature of suitably *regularised* DK models. Among these, we mention our works [12, 11] (which are the contents of Chapters 2 and 3) and also [23, 47]. With the exception of [11], all these works on regularised DK models were available prior to the publication of [34, 35].

We now turn to the stochastic TF equation (1.2). Despite the deterministic counterpart of the equation being nowadays well understood (in terms of positivity of solutions, analysis of meaningful boundary conditions and entropy estimates above all [4, 6, 7]), only a few well-posedness results are available in the stochastic case. This appears to be primarily blamed on the difficulty arising when framing relevant integral estimates for the thin-film model (such as energy and entropy estimates) in a stochastic setting. It is only recently that the concept of martingale solutions (in the declination introduced in [16, 30]) has also been used in the analysis of the stochastic TF equation with quadratic mobility (i.e., with linear noise) [24, 27].

The first work [24] considers Itô noise and mass-preserving interface potentials. The authors choose a spatial discretisation compatible with some relevant thin-film integral estimates; in addition, it gives suitable uniform a priori bounds in the application of the Itô formula to a suitable energy/entropy functional. The construction of a solution is then settled by a limit passage (in a martingale sense) which removes the discretisation and gives a solution. The specific choice of mobility (which gives the bounded derivatives of the noise) appears to be crucial for proving the necessary a priori estimates, as discussed in Chapter 4.

In [27], the authors consider Stratonovich noise, which allows for the absence of any interface potential. The discretisation is here performed in time, and according to a Trotter-Kato scheme which 'switches' from purely deterministic TF dynamics to purely stochastic TF dynamics in between consecutive time steps. Much of the a priori estimates thus rely on deterministic thin-film evolution and viscous regularisation of the stochastic thin-film dynamics: these, separately, are convenient to analyse. The limit passage recovering the martingale solution is essentially analogous to the one illustrated in [24].

1.7. Outline of the Thesis

The bulk of this thesis is contained in Chapters 2-4. Each of these chapters contains an original research paper originated from the Ph.D. work of the author.

Chapter 2 is devoted to the derivation and analysis of a regularised DK model in the case of *non-interacting* Langevin particles. We consider particles of finite rather than atomic size and, using this regularisation, we readapt and give rigour to the derivation of the original DK equation (1.1). Points of interest reside in Kolmogorov-type a priori estimates for the regularisation, covariance analysis for the regularised noise, and a well-posedness theory (in a high-probability sense) for mild solutions. The model we derive is interesting both in terms of its improved regularity properties over the original DK equation (1.1), and in terms of the aspects that are still to be understood: among these, we find the *out-of-equilibrium* regime, and the solution's almost-sure positivity.

The analysis of Chapter 2 is adapted to the case of particles weakly interacting via a pairwise potential in Chapter 3. While the general questions that we ask ourselves are the same as in Chapter 2, several technicalities related to stochastically dependent particles arise. In particular, we deploy *propagation of chaos* [46] techniques, and reformulate the a priori estimates in a suitable setting (based on Simon's compactness criterion [57]). The model we obtain accounts for meaningful nonlocal particle interactions, while being subject to the same open questions given in Chapter 2.

The issue of almost-sure positivity, which is common to both models derived in Chapters 2 and 3, is then analysed in its own right for a wider class of equations, these being non-conservative modifications to the stochastic TF equation (1.2). We provide conditions on several parameters (including mobility coefficient and source potentials) which are sufficient to extend a locally (in time) defined positive solution to any arbitrary finite time. Although we do not construct the local solution itself, this analysis sheds some light on the general positivity issue of the DK class. Our findings are consistent with the known case of quadratic mobility and Itô noise [24].

We summarise our findings, illustrate relevant future research directions, and draw our final conclusions in Chapter 5.

1.8. A heuristic overview of the scaling arguments for our regularised DK models

As previously mentioned, our regularised DK models (studied in Chapters 2 and 3) are based on Langevin systems of type (1.3), and where the particles have finite size. In addition, we will be working under the assumption that the particle size is related to the total number of particles. This fact has several important (and relatively independent) implications which are worthwhile sketching here on a heuristic level.

Proposed scaling. Let N be the number of particles, and let $\epsilon > 0$ be a parameter identifying the 'size' of a single particle via a spatial kernel w_ϵ. We relate N and ϵ with a scaling which can essentially be thought of as

$$N\epsilon^\theta = 1, \quad \text{for some } \theta > 0. \tag{1.8}$$

The larger θ is in (1.8), the larger the particles are. The choice of θ may reflect relevant physical properties of the system, such as a 'natural' particle size, or a total volume preservation.

Sketch of the argument for our regularised DK models. For any admissible pair (ϵ, N) we analyse the evolution of the ϵ-size dependent macroscopic density $\rho_\epsilon(x,t)$ and momentum density $j_\epsilon(x,t)$. We establish tightness, in the limit of N and ϵ, for the two families $\{\rho_\epsilon\}$, $\{j_\epsilon\}$, as well as for one other family of auxiliary processes (denoted by $\{j_{2,\epsilon}\}$) appearing in the evolution dynamics of $\{j_\epsilon\}$. The evolution of $\{j_\epsilon\}$ features a microscopic, finite-dimensional stochastic noise $\dot{\mathcal{Z}}_N$. We approximate $\dot{\mathcal{Z}}_N$ with a mesoscopic, infinite dimensional stochastic noise $\dot{\mathcal{Y}}_N$ using a spatial covariance comparison and some degree of information coming from the tightness argument. In addition, we relate $\{j_{2,\epsilon}\}$ to $\{\rho_\epsilon\}$ using a small temperature approximation. From this point onwards, we treat the model we have obtained, our *regularised DK model*, from a purely SPDE point of view, thus detaching the analysis from the underlying particle system. We study this SPDE in a small-noise regime.

Role of the scaling. The method sketched above benefits from (1.8) in at least three distinct points, where θ is required to exceed a threshold (which might differ from point to point). In all cases, (1.8) is required to compensate polynomial contributions (in ϵ^{-1}) mostly arising from the evaluations of relevant Sobolev-type norms associated with the kernel w_ϵ, as thoroughly explained in the following chapters. At this stage, we give a heuristic explanation for the need of these scaling applications.

Point 1: Tightness argument. The scaling (1.8) is, essentially, concerned with the hydrodynamic limit (as $N \to \infty$ and $\epsilon \to 0$) of the microscopic evolution of $\{\rho_\epsilon\}$, $\{j_\epsilon\}$. Specifically, we establish tightness of $\{\rho_\epsilon\}$, $\{j_\epsilon\}$, and $\{j_{2,\epsilon}\}$ on the level of trajectories. As our densities only depend on space, while the particle dynamics also comprises the particle velocity, the limit obtained on the macroscopic scale is not closed in the limiting density ρ and momentum density j, due to $\{j_{2,\epsilon}\}$ having a non-trivial limit j_2. As we ultimately work on the mesoscopic scale, we do not fully exploit the macroscopic limit, even though the methodology developed here proves useful later and is of independent interest. To this date, the characterisation of j_2 (also depending on the specific scaling (1.8)) is a relevant open question.

Point 2: Replacements on mesoscopic scale. A crucial part of our work involves the replacement of the microscopic stochastic noise of the particle model with a closely aligned mesoscopic one. The scaling (1.8) here 'tunes' the size of the two noises, and also determines the difference (in terms of pointwise spatial covariance) between the two noises. In essence, this scaling makes the difference of the two noises negligible w.r.t. their size, so the replacement on the mesoscopic scale is justified. Additionally, the tightness of $\{\rho_\epsilon\}$ is here also employed the reinforce the case for such a replacement. The ϵ-dependent driving force of the mesoscopic noise (denoted by $\tilde{\xi}_\epsilon$) is a space-time white noise convoluted with the kernel w_ϵ.

The methodology here reflects the one from the tightness argument, only with the focus on spatial rather than time regularity, and the scaling (1.8) determines the spatial pointwise limiting behaviour of the mesoscopic noise. In this thesis, we are interested only in a small-noise regime analysis (see below), and therefore we only consider a certain range of θ.

It is also worth mentioning that (1.8) does not currently play any role in the replacement of $j_{2,\epsilon}$ on the mesoscopic scale: this is settled by a heuristic small temperature approximation under a local equilibrium assumption.

In future works, it would be interesting to investigate other ranges of θ, possibly resulting in the survival of the noise and/or in a meaningful representation of $j_{2,\epsilon}$ in the macroscopic limit.

Point 3: Small-noise analysis of mesoscopic model. Once the replacements above are performed, we study the resulting model (*regularised DK model*) as an SPDE in its own right, thus detaching the analysis from the underlying particle model. Our main focus is to conduct a small-noise regime analysis (w.r.t. the corresponding noise-free dynamics) for the solution's maximum spatial displacement. For this purpose, we choose a norm of H^1-type: the use of this metric and the ϵ-dependent bounds for the trace of the covariance operator of $\tilde{\xi}_\epsilon$ justify the need of (1.8) for the desired small-noise regime analysis. It is worth noting that the scaling requirement here differs from that of Point 2, due the fact that we analyse the noise as a function-valued process, and not simply in a pointwise fashion.

In this part, the main open questions are mostly of analytical type, as their is no more modelling involved. Future efforts should in fact be pointed at making the scaling less restrictive, possibly by choosing a more suitable working norm or by using more accurate ϵ-dependent bounds related to $\tilde{\xi}_\epsilon$.

Chapter 2

A regularised Dean–Kawasaki model: derivation and analysis

The original DK equation (1.1) is characterised by the low regularity of its stochastic noise, whose very structure is dictated by its original derivation in physics [15]. In this chapter, we look for answers to the following three questions. Firstly, can we suitably regularise the said derivation of the DK equation in order to improve its mathematical rigour? Secondly, how 'close' is the derived regularised DK model to the original one? And thirdly, how much do we gain in terms of regularity for the resulting model?

This is joint work with Tony Shardlow and Johannes Zimmer, which was published in the *SIAM Journal on Mathematical Analysis.*

2.1. Outline of the Article

In the original derivation of the DK equation [15], a finite-size system of N Langevin particles is described by giving an equation of motion for the atomic density

$$\rho_N(x,t) := \sum_{i=1}^{N} \delta(x - q_i(t)), \qquad x \in \mathbb{R}^d, t \in [0,T], \tag{2.1}$$

where $\delta(\cdot - y)$ denotes the Dirac distribution centred at y, and $q_i(t)$ denotes the position of the i-th particle at time t. The evolution equation for ρ_N is, at least formally, given by an application of the Itô calculus applied to the composition of the distribution δ to the processes $q_i(t)$, $i = 1, \dots, N$. This evolution equation is not closed in the density ρ_N. In particular, the stochastic noise associated with the particles fluctuations is on the microscopic scale. It is thus necessary to suitably close the noise in order to achieve the representation of the particle system on the desired mesoscopic scale. It is at this stage that the distinctive DK noise term, as shown in (1.1), is proposed as a stochastically equivalent replacement to the microscopic noise. The specific nature of the noise in (1.1) is dictated by several factors, among which we find the independence of the forces driving the particles, and the definitions of relevant densities (such as ρ_N) being on the atomic scale: we will be more precise on this in due course.

The DK noise (1.1) has only been derived on a formal level: it suffers from ill-posedness in a distributional sense (what is the square root of a distribution?), and the application of the divergence operator is only formal. In addition, the analysis of the DK equation is critical, as elucidated in Subsection 1.5.3.

We provide the context and summary of the main results in Section 1. Notation and relevant assumptions for the Langevin system are given in Section 2. In particular, only stochastically *independent* particles are considered. Section 3 contains the modelling part of the work, in which the physics derivation sketched above is rigorously adapted in a function setting: there, particles are treated as having finite rather than atomic size. As a result, a regularised DK model is obtained, and its 'closedness' with the original DK model is quantified. The well-posedness of the obtained regularised DK model is investigated in Section 4.

Appendix B: Statement of Authorship

This declaration concerns the article entitled:									
A regularized Dean-Kawasaki model: derivation and analysis									
Publication status (tick one)									
draft manuscript		**Submitted**		**In review**		**Accepted**		**Published**	X
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Candidate's contribution to the paper (detailed, and also given as a percentage).	The author of the thesis has performed the bulk of the computations for this work (70%). The presentation of the contents have been shared in equal weights between all authors (33%).								
Statement from Candidate	This paper reports on original research I conducted during the period of my Higher Degree by Research candidature.								
Signed		**Date**	17.9.2019						

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A REGULARIZED DEAN–KAWASAKI MODEL: DERIVATION AND ANALYSIS*

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Abstract. The Dean–Kawasaki model consists of a nonlinear stochastic partial differential equation featuring a conservative, multiplicative, stochastic term with non-Lipschitz coefficient, driven by space-time white noise; this equation describes the evolution of the density function for a system of finitely many particles governed by Langevin dynamics. Well-posedness for the Dean–Kawasaki model is open except for specific diffusive cases, corresponding to overdamped Langevin dynamics. It was recently shown by Lehmann, Konarovskyi, and von Renesse that no regular (nonatomic) solutions exist. We derive and analyze a suitably regularized Dean–Kawasaki model of wave equation type driven by colored noise, corresponding to second-order Langevin dynamics, in one space dimension. The regularization can be interpreted as considering particles of finite size rather than describing them by atomic measures. We establish existence and uniqueness of a solution. Specifically, we prove a high-probability result for the existence and uniqueness of mild solutions to this regularized Dean–Kawasaki model.

Key words. Dean–Kawasaki model, stochastic wave equation, spatial regularization of space-time white noise, Langevin dynamics, mild solutions

AMS subject classifications. Primary, 60H15; Secondary, 35R60

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1. Introduction.

Fluctuating hydrodynamics is concerned with the description of the evolution of a large number of particles by means of suitable stochastic partial differential equations. We refer the reader to [11] and give as an example the *Dean–Kawasaki* model [8, 19]

$$\frac{\partial \rho}{\partial t}(x,t) = \underbrace{\nabla\cdot\left(\rho(x,t)\,\nabla\frac{\delta F(\rho)}{\delta\rho}\right)}_{=:\mathscr{D}} + \underbrace{\nabla\cdot\left(\sigma\sqrt{\rho(x,t)}\,\xi\right)}_{=:\mathscr{S}}. \tag{1}$$

Here $\rho\colon D\times[0,T]\subset\mathbb{R}^d\times[0,+\infty]\to[0,+\infty]$ is the density of particles, σ is a small real parameter, F is a free-energy functional, and ξ is a space-time white noise. The deterministic term $\mathscr{D}$ is a gradient-flow-driven term describing the average behavior of the system and can be derived from the Fokker–Planck analysis. The stochastic term $\mathscr{S}$ accounts for fluctuations about the mean due to the finite number of particles in the system. As a result of the divergence form, both the terms $\mathscr{D}$ and $\mathscr{S}$ account for conservation of mass in the system; see also [12, 13] for similar models.

Equation (1) poses a fascinating mathematical challenge. On one side, this equation and its more complex incarnations are widely simulated in physics; see, for example, [32, equation (59)], [24], and [10]. On the other hand, very little is known about existence and uniqueness of solutions for this class of problems, as discussed below.

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We point out three main difficulties posed by (1) from a mathematical perspective. First, the noise term $\mathscr{S}$ is defined by means of a formal divergence operator. The regularity of the argument of the divergence operator is a priori unknown. In particular, a standard $L^2(D)$-valued stochastic analysis for the argument $\sigma\sqrt{\rho(x,t)}\,\xi$ (in the sense of [29, 7], for example) would not allow us to interpret the noise $\mathscr{S}$, hence (1), in a function setting. Second, the derivation of (1) in the physics literature is formal and applicable only to empirical (thus atomic) measures. Whether a solution to (1) for smooth initial data exists is in general not clear. Third, the lack of Lipschitz continuity associated with the square root poses further difficulties.

Von Renesse and collaborators have studied regularized versions of (1) in the foundational works [33, 3, 20, 21]. They obtained existence results for measure-valued martingale solutions for modifications of (1) (in [3, 33] for the Gibbs–Boltzmann entropy functional F scaled by $\mu > 0$, and in [20] for the case $F \equiv 0$). These modifications affect the drift of (1), and they are associated with Dirichlet form arguments and with the Wasserstein geometry over the space of probability densities.

Very recently, Lehmann, Konarovskyi, and von Renesse [22] dispelled the belief that there are smooth solutions to the purely diffusive Dean–Kawasaki equation. More precisely, for (1) in one space dimension with free energy $F := \frac{N}{2}\int_D \rho(x)\log(\rho(x))\mathrm{d}x$, where (1) becomes

$$\frac{\partial\rho}{\partial t}(x,t) = \frac{N}{2}\Delta\rho(x,t) + \nabla\cdot\left(\sqrt{\rho(x,t)}\,\xi\right),$$

they showed that a unique measure-valued martingale solution exists if and only if $N \in \mathbb{N}$; in this case, the solution is the empirical distribution associated with N independent Brownian particles, so an atomic measure. The basis of this dichotomy is the interplay of the particular geometry of diffusion and noise in the context of a stochastic Wasserstein gradient flow. We also mention that a similar setting later led the authors of [22] to obtain an analogous dichotomy in the case of more general smooth drift potentials F [23].

The central differences to the approach presented below are that in [22], the underlying particle dynamics is first order (overdamped Langevin); the noise is derived from deep probabilistic arguments (describing Brownian motion in the space of probability measures with finite second moment, i.e., relying on the Wasserstein geometry); and the noise is not regularized.

The original derivation of Dean–Kawasaki equations is mathematically opaque, with one noise being replaced by a stochastically equivalent one, and with physical approximations closing the model in the density ρ under the assumption of local equilibrium (see Steps 2–3 in subsection 1.1 below); since the existence of solutions to this type of equations is so delicate, we revisit the derivation, introduce physically motivated regularizations, and then establish existence and uniqueness of solutions (in a high probability sense). The starting point is undamped (second-order) Langevin equations with on-site potential, describing the motion of finitely many particles. A key point for modeling the particles is that we do not describe them by atomic (Dirac) measures; instead, each particle is given by a Gaussian with standard deviation $\epsilon \ll 1$, centered on the particle positions. As a consequence, standard tools from stochastic calculus apply to the empirical density for N such particles. We find it useful to work with (a regularized version of) the empirical measure $\frac{1}{N}\sum_{i=1}^N \delta(x-q_i)$ and remark that both [22] and [8] use the different, but equivalent, scaling $\sum_{i=1}^N \delta(x-q_i)$; see (3) below. The advantage of the scaling chosen here is that the limit of the number of particles $N \to \infty$ is well-defined, leading to the hydrodynamic scale, and that we work in the setting of probability measures. Specifically, we study suitably combined limits of the number of particles N going to ∞ and the width parameter ϵ going to 0. Then, the noise in the resulting equations scales with $N^{-1/2}$ and disappears in the limit $N \to \infty$ (in contrast to (1); the dependency on the scaling in the deterministic and stochastic operator in (1) also plays a role in [3, 33, 22]). As in the original derivation by Dean [8], we then replace a nonclosed expression for the noise obtained by Itô calculus with a stochastically equivalent one; yet, in the framework we establish, the new noise can be compared to the original one and we obtain error bounds and show that their difference is small. In addition, we replace a nonclosed component of the deterministic drift with a closed expression by working in a low temperature regime for the Langevin system. We are then in a position to formulate, for large but finite N, a regularized stochastic wave equation of Dean–Kawasaki type. For this equation, we establish a high probability existence and uniqueness result for mild solutions using a small-noise regime analysis; more specifically, we invoke a Chebyshev inequality argument to prove that the solution stays close to a suitable deterministic process which is positive and bounded away from the non-Lipschitz noise singularity (i.e., from the identically vanishing density).

The general philosophy of this paper to derive stochastic equations describing the evolution of N Gaussians with given variance instead of N Diracs seems to be novel. Yet it seems to be natural and potentially useful in a variety of situations. For example, if one seeks to analyze the evolution of finitely many droplets in a suspension, then the description of a droplet by a Gaussian seems at least as natural as a description by a Dirac. The stochastic equation derived and studied here describes the evolution of such a system of particles. Additionally, the tightness arguments in N and ϵ developed in subsection 3.1 are of independent interest. While we use them as novel argument to compare noise expressions, they can also be useful in an alternative derivation of the hydrodynamic limit, though we do not pursue this avenue in this article.

Before describing this approach in more detail, we sketch the derivation commonly taken in the physical literature.

1.1. Original model derivation in dimension $d = 1$. The *Dean–Kawasaki* model [8, 19] arises in the mathematical description of a system of *finitely many* particles experiencing Langevin dynamics. We briefly discuss the derivation of this model by following [24, section II]. Consider N stochastically independent and identically distributed particles moving on the real line, with position and velocity $\{(q_i, p_i)\}_{i=1}^N$. More precisely, their evolution is given by the Langevin dynamics

$$\begin{cases} \dot{q}_i = p_i, \\ \dot{p}_i = (-\gamma p_i - V'(q_i)) + \sigma \dot{\beta}_i, \qquad i = 1, \dots, N, \end{cases} \tag{2}$$

starting from independent and identically distributed initial conditions $\{(q_{i,0}, p_{i,0})\}_{i=1}^N$. In (2), $\{\beta_i\}_{i=1}^N$ is a family of independent standard Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\sigma, \gamma > 0$ are given constants satisfying the fluctuation-dissipation relation $\sigma^2/(2\gamma) = k_B T_e$ (see, for example, [5]), and $V\colon \mathbb{R} \to \mathbb{R}$ is a potential. The particle system is described in terms of the global quantities

$$\rho_N(x,t) := \sum_{i=1}^N \delta(x - q_i(t)) \text{ and } j_N(x,t) := \sum_{i=1}^N p_i(t)\delta(x - q_i(t)), \qquad x \in \mathbb{R},\ t \geq 0, \tag{3}$$

representing the *local density* and the *momentum density*, respectively. These quantities, which are not rescaled in N, are to be understood in the Schwartz distribution sense, due to the presence of the Dirac distributions, denoted by δ. We sketch below how this leads to (1), the *Dean–Kawasaki* stochastic partial differential equation [8, 19], following [24].

Step 1. Evolution equations of first order in time [24, equation (4)] are derived for both ρ_N and j_N by means of standard Itô calculus, in a distributional sense. These equations are a simple superposition of the stochastic equations resulting from the Langevin dynamics (2) of each particle $i = 1, \ldots, N$. The evolution equation for ρ_N is a conservation law associated with the momentum density, and it reads $\partial\rho_N/\partial t = -\nabla \cdot j_N$. The evolution equation for j_N is, broadly speaking, an undamped equation perturbed by a particle-dependent stochastic noise.

Step 2. The aforementioned particle-dependent noise featured in the stochastic equation [24, equation (4)] associated with j_N is not of closed form (i.e., it cannot be expressed as a simple function of the quantities ρ_N and j_N). This noise is

$$\sigma \sum_{i=1}^{N} \delta(x - q_i(t))\dot{\beta}_i. \tag{4}$$

For this reason, the above noise is *formally* replaced by another noise preserving the spatial covariance structure of (4). The latter noise takes the shape

$$\sigma\sqrt{\rho_N(x,t)}\,\xi, \tag{5}$$

where ξ is a space-time white noise.

Step 3. The first-order evolution equations for ρ_N, j_N (with the noise replacement (5)) are then analyzed on the hydrodynamic scale under a local equilibrium assumption, thus giving equations in some new variables ρ and j [24, equation (11)]. In one space dimension, this system reads

$$\begin{cases} \dfrac{\partial \rho}{\partial t}(x,t) = -\dfrac{\partial j}{\partial x}(x,t), \\ \dfrac{\partial j}{\partial t}(x,t) = \left(-\gamma j(x,t) - \rho(x,t)\nabla\dfrac{\delta F(\rho)}{\delta\rho}\right) + \eta\sqrt{\rho(x,t)}\xi \end{cases} \tag{6}$$

(in suitable units, with a small parameter η), where F is a suitable free-energy functional, and δ denotes variational differentiation. The equations in (6) are then combined into a dissipative wave equation which is closed in the variable ρ [24, equation (12)]. This step provides the divergence operator for the stochastic noise of (1). The final evolution equation (1) is obtained by passing to the overdamped limit. We will not follow this last step and instead study a stochastic damped wave equation which can be seen as regularization of (6); see (9) below. For details of the procedure just sketched, we refer the reader to [24, sections IIA, IIB] and [8, 19].

1.2. Summary of the paper and main results. We now summarize the contents and main results of this paper.

We set the notation in subsection 2.1. In subsection 2.2, we define two different sets of hypotheses regarding the potential V, referred to as Assumption (G) and Assumption (NG). The first one is associated with a vanishing potential, $V \equiv 0$, which makes some specific tools of the theory of Gaussian random variables applicable. The second assumption allows for a polynomially diverging potential $V(q) \approx |q|^{2n}$, in the context of a Fokker–Planck analysis for (2).

Derivation of the regularized Dean–Kawasaki model: This is the content of section 3, and we proceed by adapting the procedure sketched in Steps 1–2, subsection 1.1, to a *function* context rather than the original distributional setting [8, 19]. We resolve the formal replacement of the noise highlighted in section 1.1 by smoothing the defining components of ρ_N and j_N. Specifically, we keep the Langevin particle system (2) and consider the *ϵ-smoothed local density* and *ϵ-smoothed momentum density*,

$$\rho_\epsilon(x,t) := \frac{1}{N}\sum_{i=1}^{N} w_\epsilon(x-q_i(t)) \text{ and } j_\epsilon(x,t) := \frac{1}{N}\sum_{i=1}^{N} p_i(t) w_\epsilon(x-q_i(t)), \quad x\in\mathbb{R},\, t\geq 0, \tag{7}$$

where $\epsilon > 0$ and $w_\epsilon(x) := (2\pi\epsilon^2)^{-1/2}\exp\{-x^2/(2\epsilon^2)\}$ is the Gaussian kernel with mean 0 and variance ϵ^2; see also Definition A.1. The kernels w_ϵ approximate the Dirac delta distribution for small values of ϵ. Notice that ρ_ϵ and j_ϵ include a rescaling in the number of particles, while ρ_N and j_N do not. Examples of realizations of ρ_ϵ are given in Figure 1.

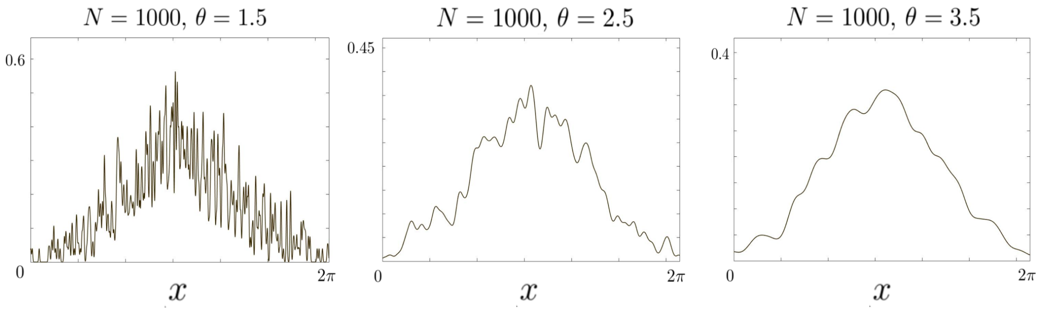

FIG. 1. *Numerical simulation of the ϵ-smoothed local density $\rho_\epsilon(\cdot,t) = N^{-1}\sum_{i=1}^N w_\epsilon(\cdot - q_i(t))$ defined in* (7)*, for a fixed time t, and on $D = [0, 2\pi]$. In this specific example, $q_i(t) \sim \mathcal{N}(\pi, 10^{0.2})$, $N = 1000$, and N and ϵ satisfy the scaling $N\epsilon^\theta = 1$ for $\theta = 1.5$ (left), $\theta = 2.5$ (middle), $\theta = 3.5$ (right). The smoothness of the density increases with θ.*

We use the ϵ-smoothed quantities (7) instead of the original quantities (3) and follow the same guidelines described in Steps 1–2 of subsection 1.1 in order to derive the regularized Dean–Kawasaki model. There, we will also consider the quantity

$$j_{2,\epsilon}(x,t) := \frac{1}{N}\sum_{i=1}^{N} p_i^2(t) w'_\epsilon(x-q_i(t)). \tag{8}$$

We do not adapt Step 3 of subsection 1.1, as we will not combine the equations for $\rho_\epsilon, j_\epsilon$ or use the hydrodynamic limit theory.

We perform the analysis of the regularized Dean–Kawasaki model both for fixed values of N and ϵ, and also by means of a simultaneous limit involving $N\to\infty$ and $\epsilon\to 0$, for N and ϵ satisfying a prescribed scaling. We first prove some preliminary uniform estimates for the three families of processes $\{\rho_\epsilon\}_\epsilon$,$\{j_\epsilon\}_\epsilon$, $\{j_{2,\epsilon}\}_\epsilon$ given in (7) and (8), as $\epsilon\to 0$. We have the following result.

PROPOSITION 1.1 (tightness of $\{\rho_\epsilon\}_\epsilon, \{j_\epsilon\}_\epsilon, \{j_{2,\epsilon}\}_\epsilon$). *Let $T > 0$, and let $D\subset\mathbb{R}$ be a bounded domain. Assume the validity of either Assumption* (G) *or Assumption* (NG)*, given below in subsection* 2.2*. Then the families of processes of $\{\rho_\epsilon\}_\epsilon, \{j_\epsilon\}_\epsilon$ are tight in $C(0,T;L^2(D))$ and $C(0,T;L^4(D))$, respectively, for $N\epsilon^\theta\geq 1$, with $\theta\geq 3$. In addition, the family $\{j_{2,\epsilon}\}_\epsilon$ is tight in $C(0,T;L^4(D))$ for $N\epsilon^\theta\geq 1$, with $\theta\geq 5$.*

Proposition 1.1 yields relative compactness in law for the families of processes $\{\rho_\epsilon\}_\epsilon, \{j_\epsilon\}_\epsilon, \{j_{2,\epsilon}\}_\epsilon$ as $\epsilon \to 0$. We show convergence for the family $\{\rho_\epsilon\}_\epsilon$ as $\epsilon \to 0$ in the following result.

Proposition 1.2. *Let $T > 0$, and let $D \subset \mathbb{R}$ be a bounded domain. Assume the validity of either Assumption* (G) *or Assumption* (NG)*, as well as the scaling $N\epsilon^\theta \geq 1$, for some $\theta \geq 3$. For each $\epsilon > 0$, let η_ϵ be the law of the process ρ_ϵ on $\mathcal{X} := C(0,T;L^2(D))$. There exists a probability measure η on $\mathcal{X}$ such that $\eta_\epsilon \overset{w}{\to} \eta$ in $\mathcal{X}$ as $\epsilon \to 0$. Here $\overset{w}{\to}$ denotes weak convergence of measures.*

The proofs of Propositions 1.1, and 1.2 under Assumption (G) are the content of subsection 3.1.

The next step, covered in subsection 3.2, is the analysis of the evolution equations for ρ_ϵ and j_ϵ, namely,

$$\left\{\begin{aligned} &\frac{\partial \rho_\epsilon}{\partial t}(x,t) = -\frac{\partial j_\epsilon}{\partial x}(x,t),\\ &\frac{\partial j_\epsilon}{\partial t}(x,t) = \left(-\gamma j_\epsilon(x,t) - j_{2,\epsilon}(x,t) - \frac{1}{N}\sum_{i=1}^N V'(q_i(t)) w_\epsilon(x-q_i(t))\right)\\ &\qquad\qquad\quad + \overbrace{\frac{\sigma}{N}\sum_{i=1}^N w_\epsilon(x-q_i(t))\dot{\beta}_i}^{=:\dot{\mathcal{Z}}_N(x,t)}, \end{aligned}\right. \tag{9}$$

where $\dot{\mathcal{Z}}_N(x,t)$ is well-defined due to regularity of w_ϵ and of the processes $\{q_i\}_{i=1}^N$. System (9) is analogous to the system of evolution equations for the original quantities ρ_N, j_N mentioned in Step 1; see [24, equation (4)].

In analogy to the original derivation of the Dean–Kawasaki model, the noise $\dot{\mathcal{Z}}_N$ is not an elementary function of ρ_ϵ and j_ϵ. For this reason, we rewrite $\dot{\mathcal{Z}}_N$ as

$$\dot{\mathcal{Z}}_N \sim \overbrace{\frac{\sigma}{\sqrt{N}}\sqrt{\rho_{\epsilon/\sqrt{2}}}\,\underbrace{Q^{1/2}_{\sqrt{2}\epsilon}\,\xi}_{=:\tilde{\xi}_\epsilon}}^{=:\dot{\mathcal{Y}}_N} + \dot{\mathcal{R}}_N, \tag{10}$$

where $\sim$ denotes equality in law, ξ is again a space-time white noise, $Q_{\sqrt{2}\epsilon}$ is the convolution operator with kernel $w_{\sqrt{2}\epsilon}$ on some spatial domain, and $\dot{\mathcal{R}}_N$ is a (small) stochastic remainder. The noise $\dot{\mathcal{Y}}_N$ is properly defined for nonnegative function ρ_ϵ. The specific structure of $\dot{\mathcal{Y}}_N$ is thoroughly discussed in subsection 3.2. We estimate the "difference" between $\dot{\mathcal{Z}}_N$ and $\dot{\mathcal{Y}}_N$ (i.e., the remainder $\dot{\mathcal{R}}_N$) with the following result.

Theorem 1.3 (error bounds for covariance structure in (9)). *Assume the validity of either Assumption* (G) *or Assumption* (NG)*. Let $D \subset \mathbb{R}$ be a bounded set, and let $T > 0$. Let N, ϵ satisfy the scaling $N\epsilon^\theta = 1$ for some fixed $\theta \geq 7/2$. Let $Q_{\sqrt{2}\epsilon} \colon L^2(D) \to L^2(D)$ be the convolution operator with kernel $w_{\sqrt{2}\epsilon}$.*

(i) *There exists $C = C(D,T)$ such that the following estimates concerning the spatial covariance of $\mathcal{Z}_N$ and $\mathcal{Y}_N$ hold for any $t \in [0,T]$ and $x_1, x_2 \in D$:*

$$\left|\mathbb{E}\left[\mathcal{Z}_N(x_1,t)\mathcal{Z}_N(x_2,t)\right] - \mathbb{E}\left[\mathcal{Y}_N(x_1,t)\mathcal{Y}_N(x_2,t)\right]\right| \leq \frac{C\sigma^2}{N} w_{\sqrt{2}\epsilon}(x_1-x_2)|x_1-x_2|^2, \tag{11}$$

$$\left|\mathbb{E}\left[\mathcal{Z}_N(x_1,t)\mathcal{Z}_N(x_2,t)\right]\right| \leq \frac{C\sigma^2}{N} w_{\sqrt{2}\epsilon}(x_1-x_2). \tag{12}$$

(ii) $\mathcal{Z}_N$ *and* $\mathcal{Y}_N$ *decay to* 0 *as* $N \to \infty$ *and* $\epsilon \to 0$. *Specifically, for any* $t \in [0,T]$ *and any* $x_1 \in D$, *we have*

$$Var\left[\mathcal{Z}_N(x_1,t)\right] \le C\epsilon^{\theta-1}, \qquad Var\left[\mathcal{Y}_N(x_1,t)\right] \le C\epsilon^{\theta-1}. \tag{13}$$

Theorem 1.3, which is proved in subsection 3.3 under Assumption (G), quantifies the error introduced when replacing the noise $\dot{\mathcal{Z}}_N$ with the multiplicative noise $\dot{\mathcal{Y}}_N$. More specifically, the bound in (11) is negligible for x_1, x_2 close to each other, when compared with the bound in (12). In addition, both $\dot{\mathcal{Z}}_N$ and $\dot{\mathcal{Y}}_N$ are negligible for distant x_1 and x_2. In combination with Proposition 1.1, Theorem 1.3 guarantees convergence of (9) to a deterministic system of equations for $N \to \infty$ and $\epsilon \to 0$. This differs from the original Dean–Kawasaki model, as we have rescaled in the number of particles N.

Remark 1.4. In the limit of infinitely many particles, $N \to \infty$, and under a local equilibrium assumption, one obtains as hydrodynamic limit (6) without the noise term and with the limit of $j_{2,\epsilon}$ being $j_2 = \nabla\frac{\delta F(\rho)}{\delta\rho}$ for a suitable F. A justification of this can be found in the analysis of the Vlasov–Fokker–Planck equation; see, for example, [26, 9]. In contrast to our setting, the Vlasov–Fokker–Planck equation is derived by relying on the empirical density defined on the combined position-momentum state space, $\tilde{\rho}_N(x,y,t) = N^{-1}\sum_{i=1}^N \delta(x-q_i(t), y-p_i(t))$. In this work, we use only the position-dependent quantities (7)–(8), as this results in a more reduced model with half the spatial dimension (i.e., position as only space variable). In addition, we do not perform the aforementioned hydrodynamic limit, but then have to close the processes $j_{2,\epsilon}$ (for fixed N) using an approximation in the context of a low temperature regime for the underlying Langevin dynamics; see subsection 3.5.

Subsection 3.4 is devoted to adapting the proofs of Proposition 1.1, Proposition 1.2, and Theorem 1.3 under Assumption (NG) instead of Assumption (G). Finally, in subsection 3.5 we give suitable approximations of the components of (9) in order to obtain expressions closed in ρ_ϵ, j_ϵ, V.

Mild solutions to the regularized Dean–Kawasaki model in a periodic setting. In section 4, we build on the contents of subsection 3.5. We work on a periodic domain, in the case of a large number of particles N. We define the *regularized Dean–Kawasaki model*

$$\left.\begin{aligned}
&\frac{\partial\rho_\epsilon}{\partial t}(x,t) = -\frac{\partial j_\epsilon}{\partial x}(x,t), \qquad x \in D = [0,2\pi],\ t\in[0,T], && \text{(14a)}\\
&\frac{\partial j_\epsilon}{\partial t}(x,t) = -\gamma j_\epsilon(x,t) - \left(\frac{\sigma^2}{2\gamma}\right)\frac{\partial\rho_\epsilon}{\partial x}(x,t) - V'_{\mathrm{per}}(x)\rho_\epsilon(x,t) && \text{(14b)}\\
&\qquad\qquad + \frac{\sigma}{\sqrt{N}}\sqrt{\rho_\epsilon(x,t)}\,\tilde{\xi}_{\mathrm{per},\epsilon},\\
&\rho_\epsilon(x,0) = \rho_0(x), \quad j_\epsilon(x,0) = j_0(x).
\end{aligned}\right\}$$

Note that in addition to the approximations made in subsection 3.5, we have also replaced $\tilde{\xi}_\epsilon$ and V with $\tilde{\xi}_{\mathrm{per},\epsilon}$ and V_{per}, the latter two being 2π-periodic versions of the former. This is a natural choice for the analysis of the equations on a periodic domain.

Remark 1.5. Equation (14) is a stochastic wave equation. Yet, standard well-posedness results for stochastic partial equations cannot be applied in a straightforward way. First, unlike the stochastic heat equation with non-Lipschitz noise coefficient [30], (14) does not have a sufficiently regular Green function associated with its linear drift operator. This results in standard semigroup techniques not being able to provide well-posedness results for (14), due to the presence of the non-Lipschitz noise in (14b).

Second, the theory of rough paths and paracontrolled distributions appears to be inapplicable, again due to the non-Lipschitz noise. Finally, the very nature of the wave equation does not seem to prevent ρ from becoming negative (e.g., a suitable maximum principle appears to be unavailable); thus it is unclear whether the noise is well-defined.

We prove various preliminary results associated with the existence theory for (14). These include the semigroup analysis associated with the deterministic integrand of (14) in subsection 4.1, a discussion on the choice of a spatially periodic noise in subsection 4.2, the analysis of the stochastic integrand of (14) in subsection 4.3, preliminary existence and uniqueness results in subsection 4.4, and a priori estimates in subsections 4.5 and 4.6. Our key result, provided in subsection 4.7, is the following.

THEOREM 1.6 (high-probability existence and uniqueness result). *Let $D = [0, 2\pi]$. Let $X_0 = (\rho_0, j_0) \in H^1_{\mathrm{per}}(D) \times H^1_{\mathrm{per}}(D)$ be a deterministic initial condition, where $H^1_{\mathrm{per}}(D)$ denotes 2π-periodic functions in $H^1(D)$. Assume that $\rho_0(x) \geq \eta$, for all $x \in D$, for some $\eta > 0$. Let the scaling $N\epsilon^\theta \geq 1$ be satisfied for some $\theta > 7$, and let $\nu \in (0, 1)$. It is possible to choose a sufficiently large number of particles N such that there exists a unique $H^1_{\mathrm{per}}(D) \times H^1_{\mathrm{per}}(D)$-valued mild solution $X_\epsilon = (\rho_\epsilon, j_\epsilon)$ satisfying* (14) *up to a time $T = T(X_0)$ on a set $F_\nu \in \mathcal{F}$ such that $\mathbb{P}(F_\nu) \geq 1 - \nu$. That is to say, the regularized Dean–Kawasaki model* (14) *is satisfied pathwise by a unique process X_ϵ on a set of probability at least $1 - \nu$.*

For the reader's convenience, we summarize how we address the three difficulties of the original Dean–Kawasaki model. First, we work in a function setting, thus the noise $\dot{\mathcal{Y}}_N$ is well-defined. Second, we do not combine the differential equations associated with ρ_ϵ (14a) and j_ϵ (14b), in contrast with [24]. On the contrary, we solve system (14) for the couple $(\rho_\epsilon, j_\epsilon)$, thus avoiding the formal application of the divergence operator for the stochastic noise of (9). Finally, we prove the above-mentioned high-probability existence and uniqueness result for (14).

The existence result of this paper is restricted to one spatial dimensional, $d = 1$. This restriction comes from Sobolev embeddings, as we point out in section 4.

Finally, Appendix A contains basic facts about Gaussian random variables, while Appendix B contains technical auxiliary results that are repeatedly used for the derivation of the regularized Dean–Kawasaki model carried out in section 3.

Remark 1.7. The assumptions of our main results (i.e., Propositions 1.1 and 1.2, and Theorems 1.3 and 1.6) are concerned with different scalings for the regularization in ϵ, namely, $N\epsilon^\theta = 1$ for some θ; see Figure 1. The lower the value of θ, the more general and less demanding the regularization is. We motivate these scalings from the specific function spaces which are involved in the proofs of the aforementioned results. In this work, we do not fully analyze the optimality of such scalings (i.e., the indentification of the lowest admissible value of θ). We limit ourselves to providing general comments on this matter in Remark 4.12.

2. Basic notation and assumptions.

2.1. Basic notation. We may use the same notation for different constants, even within the same line of computation. The dependence of a constant on given parameters will be highlighted only when it is relevant. We use the symbol $\|\cdot\|$ to denote the norm in $\mathbb{R}^d$. We use the symbol $\langle\cdot,\cdot\rangle$ to refer to the standard inner product in $\mathbb{R}^d$. For $x \in \mathbb{R}$, we define $\langle x\rangle := \sqrt{1 + x^2}$. The symbol $\mathbb{E}[X]$ denotes the expectation of an $\mathbb{R}^d$-valued random variable X defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For two $\mathbb{R}^d$-valued random variables X, Y, we denote the covariance matrix (respectively, correlation matrix) of X and Y by $\mathrm{Cov}(X,Y)$ (respectively, $\mathrm{Corr}(X,Y)$). For a real-valued random variable X, we abbreviate $\mathrm{Var}(X) := \mathrm{Cov}(X,X)$. We will use the symbol $\sim$ to indicate equivalence of laws for random variables. In particular, we write $X \sim \mathcal{N}(\mu,\sigma^2)$ for a Gaussian random variable X of mean μ and variance σ^2. We write $\mathcal{G}(y,\mu,\sigma^2)$ to denote the probability distribution function of $X \sim \mathcal{N}(\mu,\sigma^2)$, namely, $\mathcal{G}(y,\mu,\sigma^2) := (2\pi\sigma^2)^{-1/2}\exp\{-(y-\mu)^2/(2\sigma^2)\}$. Quite often, we will use the shorthand notation $w_\epsilon(y) := \mathcal{G}(y,0,\epsilon^2)$, for $\epsilon > 0$. For $X \sim \mathcal{N}(\mu,\sigma^2)$, we define its *absolute* moments $M(n,\mu,\sigma^2) := \mathbb{E}\left[|X|^n\right]$ and *plain* moments $m(n,\mu,\sigma^2) := \mathbb{E}\left[X^n\right]$ for any $n \in \mathbb{N}\cup\{0\}$. For a vector $\mu \in \mathbb{R}^d$ and a symmetric semipositive definite matrix $\Sigma \in \mathbb{R}^{d\times d}$, we write $X \sim \mathcal{N}(\mu,\Sigma)$ to denote an $\mathbb{R}^d$-valued Gaussian random vector with mean μ and covariance matrix Σ. For a domain $A \subset \mathbb{R}$, we use the standard notation $L^p(A)$ and $H^n(A)$ (for $p \in [1,\infty]$ and $n \in \mathbb{N}$) to denote the L^p-spaces on A and the Sobolev spaces of functions on A with square integrable weak derivatives up to order n. We denote n times continuously differentiable functions on A by $C^n(A)$ (for $n \in \mathbb{N}\cup\{\infty\}\cup\{0\}$).

2.2. Assumptions on the Langevin dynamics. We consider the following two different sets of assumptions associated with the Langevin dynamics (2), and in particular with the choice of potential V.

Assumption (G) (Gaussian setting for vanishing potential V). Let $T > 0$. The potential V vanishes, $V \equiv 0$. Moreover, the initial condition (q_0,p_0) to (2) is such that the solution $(q(t),p(t))$ to (2) satisfies

(i) $(q(t),p(t))$ is a bivariate Gaussian vector for all $t \in [0,T]$.
(ii) There exist $\iota > \nu > 0$ such that $\nu \le \mathit{Var}[q(t)] \le \iota$ for all $t \in [0,T]$.
(iii) The following quantities are Lipschitz on $[0,T]$: the expected values $\mu_q(t) := \mathbb{E}[q(t)]$ and $\mu_p(t) := \mathbb{E}[p(t)]$, the variances $\sigma_q^2(t) := \mathit{Var}[q(t)]$ and $\sigma_p^2(t) := \mathit{Var}[p(t)]$, and the correlation $\chi(t) := \mathit{Corr}(q(t),p(t))$.

This assumption holds generically for the Ornstein–Uhlenbeck process dynamics; see Lemma A.6.

Assumption (NG) (non-Gaussian setting for rapidly diverging $V(q) \approx |q|^{2n}$).

(i) The potential V is a $C^\infty(\mathbb{R})$-function. Furthermore, there exists $n \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$, there exists a constant C_k such that
$$\left|\frac{\partial^k V(q)}{\partial q^k}\right| \le C_k\left(1+\langle q\rangle^{2n-\min\{2,k\}}\right) \qquad \text{for all } q \in \mathbb{R}.$$
(ii) There exist two constants $C_0(V), C_1(V) > 0$ such that
$$V(q) \ge C_0^{-1}\langle q\rangle^{2n} - C_0, \qquad \left|\frac{\partial V(q)}{\partial q}\right| \ge C_1^{-1}\langle q\rangle^{2n-1} - C_1 \qquad \text{for all } q \in \mathbb{R}.$$
(iii) The joint density g_0 of the initial condition (q_0,p_0) to (2) coincides with $\overline{g}(\overline{t},q,p)$, where $\overline{t}$ is some positive time and $\overline{g}(\overline{t},q,p)$ is the solution at time $\overline{t}$ to the Fokker–Planck equation

(15)
$$\frac{\partial g}{\partial t} = -\nabla\cdot(g\mu) + \frac{\sigma^2}{2}\frac{\partial^2 g}{\partial p^2}, \qquad \mu := (p, -\gamma p - V'(q)) \qquad g(0,q,p) = \overline{g}_0(q,p),$$

started from some initial condition $\overline{g}_0 \in M^{1/2}H^{-5,-5}(\mathbb{R}^2)$. The notation $H^{s,s}(\mathbb{R}^2)$, $s > 0$, denotes the *sth*-order member of the isotropic Sobolev chain defined in [15, equation (3)], while the weight function $M(q,p) \propto \exp\{-(2\gamma/\sigma^2)\,(p^2/2 + V(q))\}$ is the Gibbs invariant measure of (15).

(iv) We have that $\lim_{q\to+\infty} V(q)/V(-q)$ exists and is finite.

Items (i) and (ii) of Assumption (NG) are slightly more restrictive than those of [15, Hypothesis 1]. In particular, we assume the potential V to diverge at infinity with no less than quadratic growth. This is encapsulated in the requirement $n \geq 1$ (instead of the requirement $n > 1/2$ made in [15, Hypothesis 1]). Item (iii) implies regularity of the initial condition g_0.

We briefly justify the choice of the above two sets of hypotheses as follows. Assumption (G) guarantees the applicability of tools inherently associated with the theory of Gaussian random variables. Then many computations can be made explicit in a relatively straightforward way. On the other hand, Assumption (NG) is more general. Our analysis under Assumption (NG) is an extension of the argument previously carried out under Assumption (G). Both these assumptions will play a role in the derivation of the regularized Dean–Kawasaki model in section 3.

3. Derivation of the regularized Dean–Kawasaki model.

We now derive the *regularized Dean–Kawasaki* model studied in this paper. In subsection 3.1, under Assumption (G), we prove a tightness result for the relevant quantities (7), (8), as well as uniqueness of the limit for the family $\{\rho_\epsilon\}_\epsilon$. These results are Propositions 1.1 and 1.2. The proof of Proposition 1.1 is nontrivial but also technical and might be skipped at a first reading. Subsection 3.2 motivates the derivation of the noise $\dot{\mathcal{Y}}_N$, which we introduced in (10). In subsection 3.3, under Assumption (G), we prove Theorem 1.3, which quantifies the difference between the noises $\dot{\mathcal{Y}}_N$ and $\dot{\mathcal{Z}}_N$ (see also (9)). In subsection 3.4 we adapt the proofs of Propositions 1.1 and 1.2 and Theorem 1.3 under Assumption (NG). Finally, subsection 3.5 gathers the relevant information from the earlier parts of section 3 in order to define a regularized Dean–Kawasaki model.

3.1. Tightness of leading quantities: Proofs of Propositions 1.1 and 1.2.

We prove some Kolmogorov-type tightness estimates for the families $\{\rho_\epsilon\}_\epsilon$, $\{j_\epsilon\}_\epsilon$, and $\{j_{2,\epsilon}\}_\epsilon$. The arguments are somewhat technical; as we are not aware of closely related results in the literature, we describe the proofs in some detail.

Proof of Proposition 1.1 *under Assumption* (G). We verify the assumption of [18, Corollary 14.9] for the families $\{\rho_\epsilon\}_\epsilon$, $\{j_\epsilon\}_\epsilon$, $\{j_{2,\epsilon}\}_\epsilon$. More specifically, for each family, we prove a suitable Kolmogorov time-regularity condition, as well as tightness of the processes at time 0.

Step 1: *Tightness of* $\{\rho_\epsilon\}_\epsilon$. We use the expansion of a square and the independence of the particles to write

$$
\begin{aligned}
&\mathbb{E}\big[\|\rho_\epsilon(\cdot,t)-\rho_\epsilon(\cdot,s)\|^2_{L^2(\mathbb{R})}\big]\\
&\quad = \frac{1}{N^2}\mathbb{E}\left[\int_{\mathbb{R}}\sum_{i,j=1}^{N}\left[w_\epsilon(x-q_i(t))-w_\epsilon(x-q_i(s))\right]\left[w_\epsilon(x-q_j(t))-w_\epsilon(x-q_j(s))\right]\mathrm{d}x\right]\\
&\quad = \frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\Big[\|w_\epsilon(\cdot-q_1(t))-w_\epsilon(\cdot-q_1(s))\|^2_{L^2(\mathbb{R})}\Big]\\
&\qquad + \frac{1}{N^2}\sum_{i\neq j}\int_{\mathbb{R}}\mathbb{E}\big[w_\epsilon(x-q_i(t))-w_\epsilon(x-q_i(s))\big]\mathbb{E}\big[w_\epsilon(x-q_j(t))-w_\epsilon(x-q_j(s))\big]\mathrm{d}x.
\end{aligned}
$$

Given the identical distribution of the particles, we deduce

$$
\begin{aligned}
&\mathbb{E}\big[\|\rho_\epsilon(\cdot,t)-\rho_\epsilon(\cdot,s)\|^2_{L^2(\mathbb{R})}\big]\\
&\quad=\frac{1}{N}\mathbb{E}\Big[\|w_\epsilon(\cdot-q_1(t))-w_\epsilon(\cdot-q_1(s))\|^2_{L^2(\mathbb{R})}\Big]\\
&\qquad+\frac{1}{N^2}\sum_{i\neq j}\big\|\mathbb{E}\big[w_\epsilon(\cdot-q_1(t))-w_\epsilon(\cdot-q_1(s))\big]\big\|^2_{L^2(\mathbb{R})}\\
&\quad\le\frac{1}{N}\underbrace{\mathbb{E}\Big[\|w_\epsilon(\cdot-q_1(t))-w_\epsilon(\cdot-q_1(s))\|^2_{L^2(\mathbb{R})}\Big]}_{=:I_1}+\underbrace{\big\|\mathbb{E}\big[w_\epsilon(\cdot-q_1(t))-w_\epsilon(\cdot-q_1(s))\big]\big\|^2_{L^2(\mathbb{R})}}_{=:\mathsf{ct}}.
\end{aligned}
\tag{16}
$$

There are two main differences between the term I_1 and the "cross-term" contribution ct. First, term I_1 is of the form $\mathbb{E}[\|\cdot\|^p_{L^p(\mathbb{R})}]$, while term ct is of the form $\|\mathbb{E}[\cdot]\|^p_{L^p(\mathbb{R})}$. Second, term ct has no decaying scaling factor in N. This means that we are forced to provide a bound for ct which is *independent* of ϵ. This bound is provided by invoking Lemmas B.2 and B.1. On the other hand, we are allowed to bound I_1 with quantities which might diverge in ϵ (these appear because of the form $\mathbb{E}[\|\cdot\|^p_{L^p(\mathbb{R})}]$, as we will point out), as long as they can be compensated by the scaling in N. These considerations are quite general, and we will apply similar reasonings at several points later on in the proof, as well as point out the relevant analogies when needed.

We occasionally drop the particle index, because of the identical distribution. We proceed to bound I_1 and ct. Using the elementary inequality

$$
1-e^{-x^2}\le x^2\quad\text{ for all } x\in\mathbb{R},\tag{17}
$$

we rewrite I_1 as

$$
\begin{aligned}
&\mathbb{E}\Big[\|w_\epsilon(\cdot-q(t))-w_\epsilon(\cdot-q(s))\|^2_{L^2(\mathbb{R})}\Big]\\
&\quad=\mathbb{E}\left[\int_{\mathbb{R}}w^2_\epsilon(x-q(t))+w^2_\epsilon(x-q(s))-2w_\epsilon(x-q(t))w_\epsilon(x-q(s))\right]\\
&\quad=\frac{1}{\sqrt{\pi\epsilon^2}}\,\mathbb{E}\left[1-\exp\left(\frac{-(q(t)-q(s))^2}{4\epsilon^2}\right)\right]\le\frac{C}{\epsilon^3}\mathbb{E}\left[|q(t)-q(s)|^2\right]\le\frac{C}{\epsilon^3}|t-s|^2,
\end{aligned}
\tag{18}
$$

where we have used Lemma A.4 and an integration in x in the last equality and (17) in the first inequality. In addition, q satisfies, by definition, the integral equation $q(t)-q(s)=\int_s^t p(z)\mathrm{d}z$. The integrability properties of p (Assumption (G)) and the Hölder inequality hence give the final inequality in (18). As for the cross-terms ct, we employ Lemma B.2, estimate (83), and then apply Lemma B.1 to deduce

$$
\begin{aligned}
&\big\|\mathbb{E}\big[w_\epsilon(\cdot-q(t))-w_\epsilon(\cdot-q(s))\big]\big\|^2_{L^2(\mathbb{R})}\\
&\quad=\int_{\mathbb{R}}\big|\mathcal{G}(x,\mu(t),\sigma_q^2(t)+\epsilon^2)-\mathcal{G}(x,\mu(s),\sigma_q^2(s)+\epsilon^2)\big|^2\mathrm{d}x\le C|t-s|^2.
\end{aligned}
$$

We combine the estimates for ct and I_1 and obtain, thanks to the prescribed scaling $N\epsilon^3\ge 1$,

$$
\mathbb{E}\big[\|\rho_\epsilon(\cdot,t)-\rho_\epsilon(\cdot,s)\|^2_{L^2(\mathbb{R})}\big]\le C\left(\frac{1}{N\epsilon^3}+1\right)|t-s|^2\le C|t-s|^2,
$$

and the time regularity is settled using Kolmogorov's continuity theorem. We now need to show that $\{\rho_\epsilon(\cdot,0)\}_\epsilon$ is tight in $L^2(D)$. We rely on the compact embedding

$H^1(D) \subset L^2(D)$ (see [2, Theorem 6.3]), and we show that $\mathbb{E}[\|\rho_\epsilon(\cdot,0)\|^2_{H^1(\mathbb{R})}]$ is uniformly bounded in ϵ. A computation analogous to (16) gives

$$\begin{aligned}
\mathbb{E}\big[\|\rho_\epsilon(\cdot,0)\|^2_{H^1(\mathbb{R})}\big] &= \mathbb{E}\big[\|\rho_\epsilon(\cdot,0)\|^2_{L^2(\mathbb{R})}\big] + \mathbb{E}\left[\left\|\frac{\partial \rho_\epsilon}{\partial x}(\cdot,0)\right\|^2_{L^2(\mathbb{R})}\right]\\
&\leq \frac{1}{N}\underbrace{\mathbb{E}\left[\int_{\mathbb{R}} w_\epsilon^2(x-q_1(0))\mathrm{d}x + w_\epsilon'^2(x-q_1(0))\mathrm{d}x\right]}_{=:I_1}\\
&\quad + \underbrace{\int_{\mathbb{R}} \mathbb{E}\big[w_\epsilon(x-q_1(0))\big]^2 + \mathbb{E}\big[w'_\epsilon(x-q_1(0))\big]^2\mathrm{d}x}_{=:\mathsf{ct}}. \qquad (19)
\end{aligned}$$

The bound $I_1 \leq C\epsilon^{-3}$ follows from Lemma A.4, in combination with the integration in x and the definition of the Gaussian moments; see Lemma A.5. The term ct can be bounded uniformly in ϵ using Lemma B.2, estimates (83) and (84). The scaling $N\epsilon^3 \geq 1$ finally implies tightness for $\{\rho_\epsilon\}_\epsilon$.

Step 2: *Tightness of* $\{j_\epsilon\}_\epsilon$. For notational convenience, we define

$$\tau_i(x,s,t) := p_i(t)w_\epsilon(x-q_i(t)) - p_i(s)w_\epsilon(x-q_i(s))$$

so that $j_\epsilon(x,t)-j_\epsilon(x,s) = N^{-1}\sum_{i=1}^N \tau_i(x,s,t)$. In the same fashion as (16), we expand

$$\begin{aligned}
&\mathbb{E}\Big[\big\|j_\epsilon(\cdot,t)-j_\epsilon(\cdot,s)\big\|^4_{L^4(\mathbb{R})}\Big] \qquad (20)\\
&\leq \frac{1}{N^3}\underbrace{\int_{\mathbb{R}}\mathbb{E}\big[\tau_1(x,s,t)^4\big]\mathrm{d}x}_{=:I_1} + \frac{C}{N^2}\underbrace{\int_{\mathbb{R}}\mathbb{E}\big[|\tau_1(x,s,t)|\big]\mathbb{E}\big[|\tau_1^3(x,s,t)|\big]\mathrm{d}x}_{=:I_2}\\
&\quad + \frac{C}{N^2}\underbrace{\int_{\mathbb{R}}\mathbb{E}\big[\tau_1^2(x,s,t)\big]^2\mathrm{d}x}_{=:I_3} + \frac{C}{N}\underbrace{\int_{\mathbb{R}}\mathbb{E}\big[\tau_1(x,s,t)\big]^2\mathbb{E}\big[\tau_1^2(x,s,t)\big]\mathrm{d}x}_{=:I_4}\\
&\quad + \underbrace{\int_{\mathbb{R}}\mathbb{E}\big[\tau_1(x,s,t)\big]^4\mathrm{d}x}_{=:\mathsf{ct}}.
\end{aligned}$$

The discussion following (16) applies analogously to the family of terms I_1, I_2, I_3, and I_4, which do contain at least one term of the form $\mathbb{E}[\tau_i(x,s,t)^p]$, and to the term ct, which is of the form $\|\mathbb{E}[\cdot]\|^p_{L^p(\mathbb{R})}$. We thus provide an ϵ-independent bound for ct and suitable ϵ-diverging bounds for I_1, I_2, I_3, and I_4.

The conditional density for bivariate Gaussian random variables, stated in Lemma A.3, implies

$$\begin{aligned}
&f_{p(t)|q(t)}(p|q(t)=b) = \mathcal{G}\left(p, \mu_p(t) + \frac{\sigma_p(t)}{\sigma_q(t)}\chi(t)(b-\mu_q(t)), (1-\chi(t)^2)\sigma_p^2(t)\right)\\
&\qquad \text{for all } b \in \mathbb{R}. \qquad (21)
\end{aligned}$$

We use the law of total expectation and (21) to compute

$$\begin{aligned}
\mathbb{E}\big[p(t)w_\epsilon(x-q(t))\big] &= \mathbb{E}\big[\mathbb{E}\big[p(t)w_\epsilon(x-q(t))|q(t)\big]\big] \qquad (22)\\
&= \mathbb{E}\left[w_\epsilon(x-q(t))\left(\mu_p(t) + \frac{\sigma_p(t)}{\sigma_q(t)}\chi(t)(q(t)-\mu_q(t))\right)\right]\\
&= a_1(t)\mathbb{E}\big[w_\epsilon(x-q(t))\big] + a_2(t)\mathbb{E}\big[w_\epsilon(x-q(t))q(t)\big],
\end{aligned}$$

where we set

$$a_1(t) := \mu_p(t) - \frac{\sigma_p(t)}{\sigma_q(t)}\chi(t)\mu_q(t), \quad a_2(t) := \frac{\sigma_p(t)}{\sigma_q(t)}\chi(t).$$

The time-dependent coefficients a_1 and a_2 are Lipschitz, thanks to Assumption (G). Keeping in mind Remark B.3, we use Lemma B.2, estimate (83), and then Lemma B.1. We deduce

$$\mathsf{ct} \le C|t-s|^{1+\beta} \tag{23}$$

for some $\beta \in (0,1)$.

We now treat the ϵ-diverging terms I_1, I_2, I_3, and I_4 in (20). By adding and subtracting the quantity $2p(t)p(s)w_{\epsilon/\sqrt{2}}(x-(q(t)+q(s))/2)$, using (17), and integrating in x, we obtain

$$\begin{aligned}\int_{\mathbb{R}} \mathbb{E}\big[\tau_1^2(x,s,t)\big]\mathrm{d}x &= \frac{1}{\sqrt{4\pi\epsilon^2}}\mathbb{E}\left[\int_{\mathbb{R}} p^2(t)w_{\frac{\epsilon}{\sqrt{2}}}(x-q(t)) + p^2(s)w_{\frac{\epsilon}{\sqrt{2}}}(x-q(s))\mathrm{d}x\right] + \\ &\quad -\frac{1}{\sqrt{4\pi\epsilon^2}}\mathbb{E}\left[\int_{\mathbb{R}} 2p(t)p(s)\exp\left\{-\frac{(q(t)-q(s))^2}{4\epsilon^2}\right\}w_{\frac{\epsilon}{\sqrt{2}}}\left(x-\frac{q(t)+q(s)}{2}\right)\mathrm{d}x\right] \\ &= \frac{1}{\sqrt{4\pi\epsilon^2}}\mathbb{E}\big[|p(t)-p(s)|^2\big] + \frac{1}{\sqrt{4\pi\epsilon^2}}\mathbb{E}\left[2p(s)p(t)\left(1-\exp\left\{-\frac{(q(t)-q(s))^2}{4\epsilon^2}\right\}\right)\right] \\ &\le \frac{1}{\sqrt{4\pi\epsilon^2}}\mathbb{E}\big[|p(t)-p(s)|^2\big] + \frac{C}{\epsilon^3}\mathbb{E}\big[2p(s)p(t)\,|q(t)-q(s)|^2\big].\end{aligned} \tag{24}$$

The first expectation in the last line of (24) satisfies $\mathbb{E}[|p(t)-p(s)|^2] \le C|t-s|$. This is implied by the Itô isometry, which we invoke because p satisfies, by definition, the stochastic integral equation $p(t)-p(s) = \int_s^t -\gamma p(z)\mathrm{d}z + \sigma\int_s^t \mathrm{d}\beta(z)$. Note the difference in time regularity with the previously discussed $\mathbb{E}[|q(t)-q(s)|^2]$; see (18). As for the second expectation in the last line of (24), we may use the Hölder inequality on the probability space to separate $p(s)p(t)$ from $|q(t)-q(s)|^2$. Using again the integrability of p granted by Assumption (G) and the Hölder inequality in time for $q(t)-q(s)$, we deduce

$$\int_{\mathbb{R}} \mathbb{E}\big[\tau_1^2(x,s,t)\big]\mathrm{d}x \le \frac{C}{\epsilon}|t-s| + \frac{C}{\epsilon^3}|t-s|^2. \tag{25}$$

In addition, we have the bound $\mathbb{E}[\tau_1(x,s,t)]^2 \le C|t-s|$, where C is independent of x and ϵ. This can be justified by relying on (22), using the fact that the right-hand side of (83) (for X being the process q) is Lipschitz in time, with Lipschitz constant independent of ϵ and x, as explained in Remark B.3. Hence, using (25), we deduce that

$$I_4 \le \frac{C}{N\epsilon^3}|t-s|^2.$$

We have completed the analysis for I_4, which is the term that requires the most care, due to the fact that it is paired with the slowest decay in N as coefficient. As for the other terms I_1, I_2, and I_3, we need not provide sharp bounds. By repeatedly applying the Hölder inequality on the probability space Ω, we deduce that I_2 and I_3

are bounded by I_1. We therefore only need to provide an estimate for I_1 in order to conclude step (ii). We write

$$\begin{aligned} I_1 &\leq C\,\mathbb{E}\Big[\int_{\mathbb{R}} (p(t)-p(s))^4 w_\epsilon^4(x-q(t))\mathrm{d}x\Big] \\ &\quad + C\,\mathbb{E}\Big[\int_{\mathbb{R}} p(s)^4 (w_\epsilon(x-q(t)) - w_\epsilon(x-q(s)))^4\mathrm{d}x\Big]. \end{aligned} \tag{26}$$

We reuse some algebraic computations from (18) to continue as

$$\begin{aligned} I_1 &\leq C\,\mathbb{E}\Big[\int_{\mathbb{R}} (p(t)-p(s))^4 w_\epsilon^4(x-q(t))\mathrm{d}x\Big] \\ &\quad + C\,\mathbb{E}\Big[\int_{\mathbb{R}} p(s)^4 (w_\epsilon(x-q(t)) - w_\epsilon(x-q(s)))^4\mathrm{d}x\Big] \\ &\leq \frac{C}{\epsilon^4}\mathbb{E}\big[(p(t)-p(s))^4\big] + C\mathbb{E}\Big[p^4(s)\frac{C}{\epsilon^2}\int_{\mathbb{R}} (w_\epsilon(x-q(t)) - w_\epsilon(x-q(s)))^2\mathrm{d}x\Big] \\ &\leq \frac{C}{\epsilon^4}\mathbb{E}\big[(p(t)-p(s))^4\big] + \frac{C}{\epsilon^2}\mathbb{E}\Big[p^4(s)\frac{1}{\epsilon}\Big(1-\exp\Big(-\frac{(q(t)-q(s))^2}{4\epsilon^2}\Big)\Big)\Big] \\ &\leq \frac{C}{\epsilon^4}\mathbb{E}\big[(p(t)-p(s))^4\big] + \frac{C}{\epsilon^5}\mathbb{E}\big[p^4(s)(q(t)-q(s))^2\big] \\ &\leq \frac{C}{\epsilon^4}|t-s|^2 + \frac{C}{\epsilon^5}\mathbb{E}\big[p^8(s)\big]^{1/2}\mathbb{E}\big[(q(t)-q(s))^4\big]^{1/2} \leq \frac{C}{\epsilon^5}|t-s|^2. \end{aligned}$$

In particular, we have used the bound $\max_y w_\epsilon(y) \leq C\epsilon^{-1}$ in the second inequality, Lemma A.4 in the third inequality, (17) in the fourth inequality, and integrability properties of p and q in the fifth and sixth inequalities. The scaling $N\epsilon^3 \geq 1$ concludes the time-regularity analysis for $\{j_\epsilon\}_\epsilon$. As for the tightness of $\{j_\epsilon(\cdot,0)\}_\epsilon$, we deal with the analogous expression of (19) for $\{j_\epsilon\}_\epsilon$. The analysis is similar, apart from the use of Lemma A.3 prior to the use of Lemma B.2 (for the corresponding term ct) and the use of the compact embedding $H^1(D) \subset L^4(D)$.

Step 3: *Tightness of* $\{j_{2,\epsilon}\}_\epsilon$. For notational convenience, we define

$$\tau_i(x,s,t) := p_i^2(t)w_\epsilon'(x-q_i(t)) - p_i^2(s)w_\epsilon'(x-q_i(s))$$

so that $j_{2,\epsilon}(x,t) - j_{2,\epsilon}(x,s) = N^{-1}\sum_{i=1}^N \tau_i(x,s,t)$. In the same fashion as (20), we expand

$$\begin{aligned} &\mathbb{E}\Big[\big\|j_{2,\epsilon}(\cdot,t) - j_{2,\epsilon}(\cdot,s)\big\|^4_{L^4(\mathbb{R})}\Big] \\ &\leq \frac{1}{N^3}\underbrace{\int_{\mathbb{R}}\mathbb{E}\big[\tau_1(x,s,t)^4\big]\mathrm{d}x}_{=:I_1} + \frac{C}{N^2}\underbrace{\int_{\mathbb{R}}\mathbb{E}\big[|\tau_1(x,s,t)|\big]\mathbb{E}\big[|\tau_1^3(x,s,t)|\big]\mathrm{d}x}_{=:I_2} \\ &\quad + \frac{C}{N^2}\underbrace{\int_{\mathbb{R}}\mathbb{E}\big[\tau_1^2(x,s,t)\big]^2\mathrm{d}x}_{=:I_3} + \frac{C}{N}\underbrace{\int_{\mathbb{R}}\mathbb{E}\big[\tau_1(x,s,t)\big]^2\mathbb{E}\big[\tau_1^2(x,s,t)\big]\mathrm{d}x}_{=:I_4} + \underbrace{\int_{\mathbb{R}}\mathbb{E}\big[\tau_1(x,s,t)\big]^4\mathrm{d}x}_{=:\mathsf{ct}}. \end{aligned} \tag{27}$$

The considerations for I_1, I_2, I_3, and I_4 and ct are analogous to the ones for the homonymous counterparts in (20). In order to estimate ct, we need to compute

$\mathbb{E}[p^2(t)w'_\epsilon(x-q(t))]$. We again rely on the conditional law (21) and the law of total expectation to write

$$\begin{aligned}&\mathbb{E}\big[p^2(t)w'_\epsilon(x-q(t))\big] = \mathbb{E}\big[\mathbb{E}\big[p^2(t)w'_\epsilon(x-q(t))|q(t)\big]\big]\\ &= \mathbb{E}\left[w'_\epsilon(x-q(t))\left\{(\mu_p(t)+\frac{\sigma_p(t)}{\sigma_q(t)}\chi(t)(q(t)-\mu_q(t)))^2+(1-\chi^2(t))\sigma_p^2(t)\right\}\right].\end{aligned} \tag{28}$$

The right-hand side of (28), thanks to Assumption (G), Lemma B.2, and Remark B.3, is of the form prescribed by Lemma B.1. Hence we deduce

$$\mathsf{ct} \le C|t-s|^{1+\beta} \qquad \text{for some } \beta>0.$$

The analysis of terms I_1, I_2, I_3, I_4 in (27) is similar to the one we carried out for the homonymous terms in (20). We set $\tilde q := (q(t)+q(s))/2$ and use Lemma A.4 to compute

$$\begin{aligned}&\int_{\mathbb{R}}\mathbb{E}\big[\tau_1^2(x,s,t)\big]\mathrm{d}x\\ &= \frac{1}{\sqrt{4\pi\epsilon^2}}\frac{1}{\epsilon^4}\left\{\mathbb{E}\left[\int_{\mathbb{R}}p^4(t)w_{\frac{\epsilon}{\sqrt2}}(x-q(t))(q(t)-x)^2+p^4(s)w_{\frac{\epsilon}{\sqrt2}}(x-q(s))(q(s)-x)^2\mathrm{d}x\right]+\right.\\ &\left.-2\mathbb{E}\left[\int_{\mathbb{R}}p^2(t)p^2(s)\exp\left\{-\frac{(q(t)-q(s))^2}{4\epsilon^2}\right\}w_{\frac{\epsilon}{\sqrt2}}(x-\tilde q)\underbrace{(q(t)-x)(q(s)-x)}_{=:T_1}\mathrm{d}x\right]\right\}.\end{aligned}$$

We add and subtract $\tilde q$ in both brackets of T_1. Similarly to the argument in (24), we rely on the x-integration with Gaussian kernels, the trivial bound $e^z\le 1$ for $z\le 0$, and we continue the above estimate,

$$\begin{aligned}&\int_{\mathbb{R}}\mathbb{E}\big[\tau_1^2(x,s,t)\big]\mathrm{d}x\\ &\le \frac{C}{\epsilon^3}\mathbb{E}\left[p^4(t)+p^4(s)-2p^2(t)p^2(s)+2p^2(t)p^2(s)\left(1-\exp\left\{-\frac{(q(t)-q(s))^2}{4\epsilon^2}\right\}\right)\right]\\ &\quad+\frac{C}{\epsilon^5}\mathbb{E}\left[\int_{\mathbb{R}}p^2(t)p^2(s)\exp\left\{-\frac{(q(t)-q(s))^2}{4\epsilon^2}\right\}|q(t)-q(s)|^2\mathrm{d}x\right]\\ &\le \frac{C}{\epsilon^3}\mathbb{E}\Big[\big|p^2(t)-p^2(s)\big|^2\Big]+\frac{C}{\epsilon^5}\mathbb{E}\big[p^2(t)p^2(s)|q(t)-q(s)|^2\big].\end{aligned} \tag{29}$$

Similarly to the argument for (24), we get

$$\int_{\mathbb{R}}\mathbb{E}\big[\tau_1^2(x,s,t)\big]\mathrm{d}x \le \frac{C}{\epsilon^3}|t-s|+\frac{C}{\epsilon^5}|t-s|^2. \tag{30}$$

Using an identical argument to the proof concerning $\{j_\epsilon\}_\epsilon$, we have that $\mathbb{E}[\tau_1(x,s,t)]^2\le C|t-s|$, where C is independent of x and ϵ. In combination with (30), this yields

$$I_4 \le \frac{C}{\epsilon^5}|t-s|^2.$$

By repeatedly applying the Hölder inequality on the probability space Ω, we deduce that I_2, I_3 are bounded by I_1. We therefore only need to provide an estimate for I_1 in order to conclude step (iii). We write

$$
\begin{aligned}
I_1 \leq{}& C\,\mathbb{E}\bigg[\int_{\mathbb{R}} (p^2(t)-p^2(s))^4 w_\epsilon'^4(x-q(t))\mathrm{d}x\bigg]\\
&+ C\,\mathbb{E}\bigg[\int_{\mathbb{R}} p(s)^8 (w_\epsilon'(x-q(t)) - w_\epsilon'(x-q(s)))^4\mathrm{d}x\bigg].
\end{aligned}
\tag{31}
$$

We notice that $\max_y |w_\epsilon'(y)| \leq C\epsilon^{-2}$. We rely on some computations in (29) and bound I_1 as

$$
\begin{aligned}
I_1 \leq{}& C\,\mathbb{E}\bigg[\int_{\mathbb{R}} (p^2(t)-p^2(s))^4 w_\epsilon'^4(x-q(t))\mathrm{d}x\bigg]\\
&+ C\,\mathbb{E}\bigg[\int_{\mathbb{R}} p(s)^8 (w_\epsilon'(x-q(t)) - w_\epsilon'(x-q(s)))^4\mathrm{d}x\bigg]\\
\leq{}& \frac{C}{\epsilon^8}\mathbb{E}\big[(p(t)-p(s))^4(p(t)+p(s))^4\big]\\
&+ \frac{C}{\epsilon^4}\,\mathbb{E}\bigg[p^8(s)\int_{\mathbb{R}} \big|w_\epsilon'(x-q(t)) - w_\epsilon'(x-q(s))\big|^2\mathrm{d}x\bigg]\\
\leq{}& \frac{C}{\epsilon^8}\mathbb{E}\big[(p(t)-p(s))^4(p(t)+p(s))^4\big] + \frac{C}{\epsilon^4}\,\mathbb{E}\bigg[\frac{C}{\epsilon^5}p^8(s)|q(t)-q(s)|^2\bigg]\\
\leq{}& \frac{C}{\epsilon^8}\mathbb{E}\big[(p(t)-p(s))^8\big]^{1/2}\mathbb{E}\big[(p(t)+p(s))^8\big]^{1/2}\\
&+ \frac{C}{\epsilon^9}\mathbb{E}\big[p^{16}(s)\big]^{1/2}\mathbb{E}\big[|q(t)-q(s)|^4\big]^{1/2} \leq \frac{C}{\epsilon^9}|t-s|^{1+\beta},
\end{aligned}
$$

where we have also used the Burkholder–Davis–Gundy inequality to estimate $\mathbb{E}[(p(t)-p(s))^8]$. The required time regularity is established. As for the tightness of $\{j_{2,\epsilon}(\cdot,0)\}_\epsilon$, we can deal with the analogous expression of (19) for $\{j_{2,\epsilon}\}_\epsilon$. The analysis is similar, apart from the use of Lemma A.3 prior to the use of Lemma B.2 (for the corresponding term ct) and the use of the compact embedding $H^1(D) \subset L^4(D)$. ◻

Remark 3.1. The scaling N^{-1} involved in the definitions of ρ_ϵ and j_ϵ is crucial for the tightness for $\{\rho_\epsilon\}_\epsilon$, $\{j_\epsilon\}_\epsilon$, and $\{j_{2,\epsilon}\}_\epsilon$. This scaling differs from the original Dean–Kawasaki derivation with nonrescaled leading quantities (3).

Remark 3.2. The scaling (of ϵ and N) associated with the family $\{j_{2,\epsilon}\}_\epsilon$ is more restrictive than the one associated with the family $\{j_\rho\}_\epsilon$; this is due to the need to estimate quantities related to derivatives of the kernel w_ϵ. The different hypotheses on θ are justified by the computations associated with term I_1 (in the case of $\{\rho_\epsilon\}_\epsilon$) and by the computations associated with term I_4 (in the case of $\{j_\epsilon\}_\epsilon$ and $\{j_{2,\epsilon}\}_\epsilon$). The scalings of Proposition 1.1 are compatible with the assumptions of our key result, Theorem 1.6.

Proof of Proposition 1.2 *under Assumption* (G). Prohorov's theorem [18, Theorem 14.3] and Proposition 1.1 imply weak convergence up to subsequences for the family $\{\eta_\epsilon\}_\epsilon$ in $\mathcal{X}$ as $\epsilon \to 0$. In order to conclude the proof, we need to prove uniqueness of the weak limit η. Let us take two sequences $\{(a_n, N_n^a)\}_n$ and $\{(b_n, N_n^b)\}_n$ satisfying the scaling prescribed in the hypothesis and such that $\eta_{a_n} \overset{w}{\to} \eta_1$ and $\eta_{b_n} \overset{w}{\to} \eta_2$ in $\mathcal{X}$. In order to show that $\eta_1 = \eta_2$, we just need to show that the finite-dimensional laws coincide; see [18, Proposition 2.2]. Let π be a projection from $\mathcal{X}$ onto a finite but arbitrary number of times $0 \leq t_1 \leq \cdots \leq t_m \leq T$. Take a bounded Lipschitz function $g\colon X^m := [L^2(D)]^m \to \mathbb{R}$. Then

$$\left|\int_{\mathcal{X}} g(\pi(p))\mathrm{d}\eta_{a_n}(p) - \int_{\mathcal{X}} g(\pi(p))\mathrm{d}\eta_{b_n}(p)\right|^2 = \left|\mathbb{E}\big[g(\pi(\rho_{a_n}))\big] - \mathbb{E}\big[g(\pi(\rho_{b_n}))\big]\right|^2$$
$$\leq L(g)\mathbb{E}\Big[\|\pi(\rho_{a_n}) - \pi(\rho_{b_n})\|_{[L^2(D)]^m}\Big]^2 \leq L(g)\sum_{j=1}^{m}\mathbb{E}\Big[\int_{\mathbb{R}} (\rho_{a_n}(x,t_j) - \rho_{b_n}(x,t_j))^2\,\mathrm{d}x\Big], \tag{32}$$

where we have used the Hölder inequality in the last step. Let us denote $N_n^M := \max\{N_n^a; N_n^b\}$ and $N_n^m := \min\{N_n^a; N_n^b\}$. For each $j \in \{1,\ldots,m\}$, we expand the square of the sum of $N_n^a + N_n^b$ terms in the jth term of (32). As N_n^a and N_n^b might differ, it is convenient to split the resulting $(N_n^a + N_n^b)^2$ product terms into six different categories. We have

- N_n^a terms of type $(N_n^a)^{-2}w_{a_n}^2(x - q_i(t_j))$,
- N_n^b terms of type $(N_n^b)^{-2}w_{b_n}^2(x - q_i(t_j))$,
- $2N_n^m$ terms of type $-(N_n^M N_n^m)^{-1}w_{a_n}(x - q_i(t_j))w_{b_n}(x - q_i(t_j))$,
- $N_n^a(N_n^a - 1)$ terms of type $(N_n^a)^{-2}w_{a_n}(x - q_i(t_j))w_{a_n}(x - q_k(t_j))$, where $i \neq k$,
- $N_n^b(N_n^b - 1)$ terms of type $(N_n^b)^{-2}w_{b_n}(x - q_i(t_j))w_{b_n}(x - q_k(t_j))$, where $i \neq k$,
- $2N_n^M N_n^m - 2N_n^m$ terms of type $-(N_n^M N_n^m)^{-1}w_{a_n}(x - q_i(t_j))w_{b_n}(x - q_k(t_j))$, where $i \neq k$.

With the help of Lemma A.4 and the scaling of $\{(a_n, N_n^a)\}_n$ and $\{(b_n, N_n^b)\}_n$, we deduce that the contributions of the first three families to the right-hand side of (32) vanish in the limit $n \to \infty$. The contribution of the remaining three families is given by

$$\sum_{j=1}^{m}\Bigg\{\frac{N_n^a(N_n^a-1)}{(N_n^a)^2}\mathbb{E}\big[w_{\sqrt{2}a_n}(q_1(t_j) - q_2(t_j))\big] + \frac{N_n^b(N_n^b-1)}{(N_n^b)^2}\mathbb{E}\big[w_{\sqrt{2}b_n}(q_1(t_j) - q_2(t_j))\big]$$
$$- \frac{2N_n^M N_n^m - 2N_n^m}{N_n^M N_n^m}\mathbb{E}\Big[w_{\sqrt{a_n^2+b_n^2}}(q_1(t_j) - q_2(t_j))\Big]\Bigg\}. \tag{33}$$

The probability density functions of the random variables $q_1(t_j) - q_2(t_j)$, $j = 1,\ldots,m$, which we denote by $f_{q_1(t_j)-q_2(t_j)}$, belong to the Schwartz space $\mathcal{S}$ (i.e., the space of rapidly decaying real-valued functions on $\mathbb{R}$). This can be justified as follows. The density of the sum of two continuous independent real-valued random variables is given by the convolution of the densities of the two random variables. In addition, for $f_1, f_2 \in \mathcal{S}$ we have that also $f_1 * f_2 \in \mathcal{S}$. As a consequence of Assumption (G), the laws of $q_1(t_j)$ and $-q_2(t_j)$, $j = 1,\ldots,m$, are Gaussian, and hence they belong to $\mathcal{S}$. We can then rewrite the expectations in (33) with dualities in $\mathcal{S}'$, and we deduce the convergence of the jth term of the sum to

$$f_{q_1(t_j)-q_2(t_j)}(0) + f_{q_1(t_j)-q_2(t_j)}(0) - 2f_{q_1(t_j)-q_2(t_j)}(0) = 0, \qquad j \in \{1,\ldots,m\},$$

by means of the convergence $w_\epsilon \to \delta$ in $\mathcal{S}'$ for $\epsilon \to 0$. This leads to

$$\int_{X^m} g(z)\mathrm{d}(\pi_*\eta_1)(z) = \lim_{n\to\infty}\int_{X^m} g(z)\mathrm{d}(\pi_*\eta_{a_n})(z)$$
$$= \lim_{n\to\infty}\int_{X^m} g(z)\mathrm{d}(\pi_*\eta_{b_n})(z) = \int_{X^m} g(z)\mathrm{d}(\pi_*\eta_2)(z),$$

where π_* indicates a push-forward of measures by π. Uniqueness of weak limits implies that $\pi_*\eta_1$ and $\pi_*\eta_2$ (the projections of η_1 and η_2 onto $\{t_1,\ldots,t_m\}$) coincide. Since the times involved are arbitrary, we deduce $\eta_1 \equiv \eta_2$. This concludes the proof. ∎

3.2. Noise replacement in evolution system for $(\rho_\epsilon, j_\epsilon)$. We now replicate the analysis described in Steps 1–2 of subsection 1.1 adapted to the setting considered here, in order to derive a regularized Dean–Kawasaki model. It is straightforward to derive system (9) using the Itô calculus on ρ_ϵ and j_ϵ. System (9) is similar to the system of evolution equations for the original quantities ρ_N and j_N; see [24, equation (4)]. In particular, in analogy to the original derivation of the Dean–Kawasaki model, the noise term $\dot{\mathcal{Z}}_N = \sigma N^{-1}\sum_{i=1}^N w_\epsilon(x-q_i(t))\dot{\beta}_i$ is not a closed expression of the leading quantities ρ_ϵ and j_ϵ. For this reason, we replace $\dot{\mathcal{Z}}_N$ with a multiplicative noise, which we initially take to be of the form

$$\frac{\sigma}{\sqrt{N}} f(\rho_\epsilon) Q_\epsilon^{1/2}\xi, \tag{34}$$

where ξ is a space-time white noise, $f\colon \mathbb{R}\to\mathbb{R}$ is to be determined, and Q_ϵ is a suitable spatial operator to be determined as well. In order to understand the above chosen structure, we first compute the spatial covariance for $\mathcal{Z}_N$. For given points $x_1, x_2 \in \mathbb{R}$, we have

$$\begin{aligned}
&\mathbb{E}\big[\mathcal{Z}_N(x_1,t)\mathcal{Z}_N(x_2,t)\big]\\
&\quad= \mathbb{E}\left[\left(\int_0^t \frac{\sigma}{N}\sum_{i=1}^N w_\epsilon(x_1-q_i(u))\mathrm{d}\beta_i(u)\right)\left(\int_0^t \frac{\sigma}{N}\sum_{i=1}^N w_\epsilon(x_2-q_i(u))\mathrm{d}\beta_i(u)\right)\right]\\
&\quad= \frac{\sigma^2}{N^2}\mathbb{E}\left[\sum_{i=1}^N\left(\int_0^t w_\epsilon(x_1-q_i(u))\mathrm{d}\beta_i(u)\right)\left(\int_0^t w_\epsilon(x_2-q_i(u))\mathrm{d}\beta_i(u)\right)\right]\\
&\qquad+ \frac{\sigma^2}{N^2}\mathbb{E}\left[\sum_{i\neq j}\left(\int_0^t w_\epsilon(x_1-q_i(u))\mathrm{d}\beta_i(u)\right)\left(\int_0^t w_\epsilon(x_2-q_j(u))\mathrm{d}\beta_j(u)\right)\right]\\
&\quad= \frac{\sigma^2}{N^2}\mathbb{E}\left[\sum_{i=1}^N\int_0^t w_\epsilon(x_1-q_i(u))w_\epsilon(x_2-q_i(u))\mathrm{d}u\right],
\end{aligned}$$

where in the last equality we have used basic Itô calculus, as well as the fact that stochastic integrals driven by independent noises are uncorrelated. Lemma A.4 gives $w_\epsilon(x_1-q_i(u))w_\epsilon(x_2-q_i(u)) = w_{\sqrt{2}\epsilon}(x_1-x_2)w_{\epsilon/\sqrt{2}}(q_i(u)-(x_1+x_2)/2)$ for all $i=1,\dots,N$. By summing over $i=1,\dots,N$ and dividing by N, we conclude that

$$N^{-1}\sum_{i=1}^N w_\epsilon(x_1-q_i(u))w_\epsilon(x_2-q_i(u)) = w_{\sqrt{2}\epsilon}(x_1-x_2)\rho_{\epsilon/\sqrt{2}}((x_1+x_2)/2,u).$$

We deduce

$$\mathbb{E}\big[\mathcal{Z}_N(x_1,t)\mathcal{Z}_N(x_2,t)\big] = w_{\sqrt{2}\epsilon}(x_1-x_2)\int_0^t \mathbb{E}\left[\frac{\sigma^2}{N}\rho_{\epsilon/\sqrt{2}}\left(\frac{x_1+x_2}{2},u\right)\right]\mathrm{d}u. \tag{35}$$

Equation (35) indicates how to define the multiplicative noise (34). The term $w_{\sqrt{2}\epsilon}(x_1-x_2)$ is deterministic. It is then not unreasonable to assume that such a term can be associated with the covariance structure for the stochastic noise in (34). On the other hand, the random variable in the right-hand side of (35) should, according to Itô calculus, be the square of the stochastic integrand of (34) evaluated at $(x_1+x_2)/2$. We thus propose the following noise replacement for $\mathcal{Z}_N$:

$$\dot{\mathcal{Y}}_N := \frac{\sigma}{\sqrt{N}}\sqrt{\rho_{\epsilon/\sqrt{2}}}\,\underbrace{Q_{\sqrt{2}\epsilon}^{1/2}\xi}_{\tilde{\xi}_\epsilon},$$

where $Q_{\sqrt{2}\epsilon}$ is a convolution operator with kernel $w_{\sqrt{2}\epsilon}$. The domain of such an operator is specified in the statement of Theorem 1.3, whose proof is provided in the next subsection.

Remark 3.3. Note that $\tilde{\xi}_\epsilon$ is a spatially correlated noise approximating the action of a space-time white noise for small values of ϵ. Also note the scaling $\epsilon/\sqrt{2}$, as opposed to the original scaling ϵ, characterizing $\rho_{\epsilon/\sqrt{2}}$ in the definition of noise $\dot{\mathcal{Y}}_N$. The factor $\sqrt{2}$ appears for simple analytical reasons. This will not affect our considerations for the limit $\epsilon \to 0$, $N \to \infty$, as we will point out in subsection 3.5.

3.3. Covariance error bound associated with noise replacement. The main modeling result concerns a thorough comparison of the stochastic noises $\dot{\mathcal{Z}}_N$ and the noise $\dot{\mathcal{Y}}_N$ just introduced. Specifically, we estimate the "price" one has to pay in order to replace $\mathcal{Z}_N$ with $\mathcal{Y}_N$ in (9). More specifically, we are interested in quantifying the size of $\mathcal{R}_N = \mathcal{Z}_N - \mathcal{Y}_N$ and $\mathcal{Y}_N$ in terms of ϵ, N. Our goal is to prove that, in the limit of $\epsilon \to 0$ and $N \to \infty$, the remainder $\mathcal{R}_N$ is negligible with respect to $\mathcal{Y}_N$. As a consequence, exchanging the stochastic noises results in a negligible correction.

Proof of Theorem 1.3 *under Assumption* (G). The convolution operator $Q_{\sqrt{2}\epsilon}$ is defined as $Q_{\sqrt{2}\epsilon}\colon L^2(D) \to L^2(D)\colon f \mapsto Q_{\sqrt{2}\epsilon}f(\cdot) := \int_D w_{\sqrt{2}\epsilon}(\cdot - y)f(y)\mathrm{d}y$. We compare the noises $\mathcal{Z}_N$, $\mathcal{Y}_N$ by means of their spatial covariance structures at any given time $t \in [0,T]$, for any couple of points $x_1, x_2 \in D$. Following on the construction in the previous section, we have

$$\mathbb{E}\big[\mathcal{Z}_N(x_1,t)\mathcal{Z}_N(x_2,t)\big] = \frac{\sigma^2}{N} w_{\sqrt{2}\epsilon}(x_1-x_2)\int_0^t \mathbb{E}\Big[\rho_{\epsilon/\sqrt{2}}\Big(\frac{x_1+x_2}{2},s\Big)\Big]\mathrm{d}s,$$

and with similar arguments one finds

$$\mathbb{E}\big[\mathcal{Y}_N(x_1,t)\mathcal{Y}_N(x_2,t)\big] = \frac{\sigma^2}{N} w_{\sqrt{2}\epsilon}(x_1-x_2)\int_0^t \mathbb{E}\Big[\sqrt{\rho_{\epsilon/\sqrt{2}}(x_1,s)\rho_{\epsilon/\sqrt{2}}(x_2,s)}\Big]\mathrm{d}s.$$

We notice that the two covariances share the common prefactor $\sigma^2 N^{-1} w_{\sqrt{2}\epsilon}(x_1-x_2)$. Our analysis will thus be focused on the terms where the two expressions differ. If we want to evaluate the difference of the two above covariance expressions, it is useful to study, for any given time $s \in [0,t]$,

$$\mathbb{E}\Big[\Big|\rho_{\epsilon/\sqrt{2}}\Big(\frac{x_1+x_2}{2},s\Big) - \sqrt{\rho_{\epsilon/\sqrt{2}}(x_1,s)\rho_{\epsilon/\sqrt{2}}(x_2,s)}\Big|\Big]. \tag{36}$$

For notational convenience, we define $m := (x_1+x_2)/2$ and drop the time dependence for $\rho_{\epsilon/\sqrt{2}}$. We add and subtract $\rho_{\epsilon/\sqrt{2}}(m)$ to both $\rho_{\epsilon/\sqrt{2}}(x_1)$ and $\rho_{\epsilon/\sqrt{2}}(x_2)$. As a result, the random variable in (36) turns into

$$\Big|\rho_{\epsilon/\sqrt{2}}(m) - \sqrt{\rho^2_{\epsilon/\sqrt{2}}(m) + b(x_1,x_2)}\Big| = \Big|\rho_{\epsilon/\sqrt{2}}(m)\Big|\left(1 - \sqrt{1 + \frac{b(x_1,x_2)}{\rho^2_{\epsilon/\sqrt{2}}(m)}}\right) \le \frac{|b(x_1,x_2)|}{\rho_{\epsilon/\sqrt{2}}(m)},$$

where we have defined

$$\begin{aligned} b(x_1,x_2) &:= \rho_{\epsilon/\sqrt{2}}(m)\Big[\rho_{\epsilon/\sqrt{2}}(x_1) + \rho_{\epsilon/\sqrt{2}}(x_2) - 2\rho_{\epsilon/\sqrt{2}}(m)\Big] \\ &\quad + (\rho_{\epsilon/\sqrt{2}}(x_1) - \rho_{\epsilon/\sqrt{2}}(m))(\rho_{\epsilon/\sqrt{2}}(x_2) - \rho_{\epsilon/\sqrt{2}}(m)). \end{aligned}$$

We can thus bound the random variable in (36) by the sum

$$
\begin{aligned}
&\left|\rho_{\epsilon/\sqrt{2}}(x_1)+\rho_{\epsilon/\sqrt{2}}(x_2)-2\rho_{\epsilon/\sqrt{2}}(m)\right| \\
&\quad+\frac{\left|\rho_{\epsilon/\sqrt{2}}(x_1)-\rho_{\epsilon/\sqrt{2}}(m)\right|\left|\rho_{\epsilon/\sqrt{2}}(x_2)-\rho_{\epsilon/\sqrt{2}}(m)\right|}{\rho_{\epsilon/\sqrt{2}}(m)} =: T_1+T_2. \qquad (37)
\end{aligned}
$$

Expected value of term T_2. We use the Hölder inequality twice and we obtain

$$
\mathbb{E}[T_2] \le \mathbb{E}\Big[\rho^{-2}_{\epsilon/\sqrt{2}}(m)\Big]^{\frac12}\mathbb{E}\Big[\Big|\rho_{\epsilon/\sqrt{2}}(x_1)-\rho_{\epsilon/\sqrt{2}}(m)\Big|^4\Big]^{\frac14}\mathbb{E}\Big[\Big|\rho_{\epsilon/\sqrt{2}}(x_2)-\rho_{\epsilon/\sqrt{2}}(m)\Big|^4\Big]^{\frac14}. \qquad (38)
$$

The first expectation in the right-hand side of (38) can be bounded, independently of N,ϵ, by means of Proposition B.8. The two remaining expectations in (38) are identical up to a swap of x_1 and x_2, hence we analyze just one of them.

In analogy to some computations previously carried out for (20) and (27), we set $\tau(x_1,m):=w_\epsilon(x_1-q_1(s))-w_\epsilon(m-q_1(s))$. We expand

$$
\begin{aligned}
\mathbb{E}\Big[\Big|\rho_{\epsilon/\sqrt{2}}(x_1)-\rho_{\epsilon/\sqrt{2}}(m)\Big|^4\Big] \le{}& \frac{1}{N^3}\underbrace{\mathbb{E}\big[\tau^4(x_1,m)\big]}_{=:I_1}+\frac{C}{N^2}\underbrace{\mathbb{E}\big[|\tau(x_1,m)|\big]\mathbb{E}\big[\big|\tau^3(x_1,m)\big|\big]}_{=:I_2} \\
&+\frac{C}{N^2}\underbrace{\mathbb{E}\big[\tau^2(x_1,m)\big]^2}_{=:I_3}+\frac{C}{N}\underbrace{\mathbb{E}\big[\tau(x_1,m)\big]^2\mathbb{E}\big[\tau^2(x_1,m)\big]}_{=:I_4} \\
&+\underbrace{\mathbb{E}\big[\tau(x_1,m)\big]^4}_{=:\mathrm{ct}}. \qquad (39)
\end{aligned}
$$

Note the absence of integration in x, as opposed to (20) and (27). We use Lemma B.2 and a first-order Taylor approximation in space together with Assumption (G)(ii) to deduce

$$
\big|\mathbb{E}\big[\tau(x_1,m)\big]\big| = \big|\mathcal{G}(x_1,\mu_q(s),\sigma_q^2(s)+\epsilon^2)-\mathcal{G}(m,\mu_q(s),\sigma_q^2(s)+\epsilon^2)\big| \le C|x_1-x_2|.
$$

We rely on Lemmas A.4 and B.2 to write $\mathbb{E}[\tau^2(x_1,m)]$ as

$$
\begin{aligned}
&\frac{1}{\sqrt{4\pi\epsilon^2}}\Big[w_{\epsilon/\sqrt{2}}(x_1-q_1(s))+w_{\epsilon/\sqrt{2}}(m-q_1(s)) \\
&\qquad -2w_{\epsilon/\sqrt{2}}\Big[\frac{x_1+m}{2}-q_1(s)\Big]\exp\Big\{-\frac{(x_1-m)^2}{4\epsilon^2}\Big\}\Big] \\
&\quad=\frac{1}{\sqrt{4\pi\epsilon^2}}\big\{\mathcal{G}(x_1,\mu_q(s),\sigma_q^2(s)+\epsilon^2/2)+\mathcal{G}(m,\mu_q(s),\sigma_q^2(s)+\epsilon^2/2) \\
&\qquad +2\mathcal{G}\Big(\frac{x_1+m}{2},\mu_q(s),\sigma_q^2(s)+\epsilon^2/2\Big)\Big\} \\
&\qquad +\frac{2}{\sqrt{4\pi\epsilon^2}}\mathcal{G}\Big(\frac{x_1+m}{2},\mu_q(s),\sigma_q^2(s)+\epsilon^2/2\Big)\Big\{1-\exp\Big\{-\frac{(x_1-m)^2}{4\epsilon^2}\Big\}\Big\}.
\end{aligned}
$$

We use a second-order approximation of the type $|f(x_1)+f(m)-2f((x_1+m)/2)| \le C|x_1-m|^2$ applied to $f(x)=\mathcal{G}(x,\mu_q(s),\sigma_q^2(s)+\epsilon^2/2)$, as well as inequality (17), to deduce

$$\mathbb{E}\big[\tau^2(x_1,m)\big] \le C\left(\frac{1}{\epsilon}+\frac{1}{\epsilon^3}\right)|x_1-x_2|^2 \le \frac{C}{\epsilon^3}|x_1-x_2|^2. \tag{40}$$

The bound $\max_y |w'_\epsilon(y)| \le C\epsilon^{-2}$, the mean-value theorem, and (40) allow us to deduce

$$\mathbb{E}\big[\tau^4(x_1,m)\big] \le \frac{C}{\epsilon^4}|x_1-x_2|^2\mathbb{E}\big[\tau^2(x_1,m)\big] \le \frac{C}{\epsilon^7}|x_1-x_2|^4.$$

The above estimate is the most demanding in terms of the scaling N,ϵ and justifies the hypothesis $\theta \ge 7/2$. Finally, the terms $\mathbb{E}[|\tau^3(x_1,m)|], \mathbb{E}[|\tau(x_1,m)|]$ can be bounded, by means of the Hölder inequality, by $\mathbb{E}[\tau^4(x_1,m)]^{3/4}$ and $\mathbb{E}[\tau^4(x_1,m)]^{1/4}$, respectively. We can put all these estimates together for the benefit of I_1, I_2, I_3, I_4, and $\mathtt{ct}$ in (39) and obtain

$$\mathbb{E}\left[\left|\rho_{\epsilon/\sqrt{2}}(x_1)-\rho_{\epsilon/\sqrt{2}}(m)\right|^4\right] \le C|x_1-x_2|^4.$$

The estimate for points x_2 and m replacing x_1 and m is identical. As a result of the above observations, we can bound the left-hand-side in (38), thus obtaining

$$T_2 \le C|x_1-x_2|^2 \tag{41}$$

for C independent of N and ϵ.

Expected value of term T_1. Using similar arguments to the analysis of T_2, it is not difficult to show that

$$\begin{aligned}&\mathbb{E}\left[\left|\rho_{\epsilon/\sqrt{2}}(x_1)+\rho_{\epsilon/\sqrt{2}}(x_2)-2\rho_{\epsilon/\sqrt{2}}(m)\right|\right]\\ &\quad\le \mathbb{E}\left[\left|\rho_{\epsilon/\sqrt{2}}(x_1)+\rho_{\epsilon/\sqrt{2}}(x_2)-2\rho_{\epsilon/\sqrt{2}}(m)\right|^2\right]^{1/2}\\ &\quad\le C|x_1-x_2|^2\end{aligned}$$

by using a fourth-order approximation of the type $|f(x_1)+f(x_2)+6f(m)-4f(m_1)-4f(m_2)| \le C|x_1-x_2|^4$, where $x_1<m_1<m<m_2<x_2$ are equidistanced. We skip the details. We combine the estimates for T_1 and T_2 and deduce

$$\left|\mathbb{E}\big[\mathcal{Z}_N(x_1,t)\mathcal{Z}_N(x_2,t)\big]-\mathbb{E}\big[\mathcal{Y}_N(x_1,t)\mathcal{Y}_N(x_2,t)\big]\right| \le \frac{C\sigma^2}{N}w_{\sqrt{2}\epsilon}(x_1-x_2)|x_1-x_2|^2,$$

which is exactly (11). Using Lemma B.1, it is also immediate to notice that

$$\left|\mathbb{E}\big[\mathcal{Z}_N(x_1,t)\mathcal{Z}_N(x_2,t)\big]\right| \le \frac{C\sigma^2}{N}w_{\sqrt{2}\epsilon}(x_1-x_2),$$

which is (12), and the proof of Theorem 1.3(i) is complete. The proof of (ii) is a straightforward consequence of the estimate $N^{-1}w_{\sqrt{2}\epsilon}(x_1,x_2) \le \epsilon^{\theta-1}$ and of (11), (12). ◻

Remark 3.4. The proof of Theorem 1.3 employs a multiplicative approach for the estimation of the random variable in (36). We rely on the estimate $\left|\sqrt{a^2}-\sqrt{a^2+c}\right| \le |c/a|$, instead of using the standard estimate

$$\left|\sqrt{a^2}-\sqrt{a^2+c}\right| \le \sqrt{|c|}. \tag{42}$$

In our specific case, we have $a := \rho_{\epsilon/\sqrt{2}}(m)$ and $c := b(x_1,x_2)$. The multiplicative approach has the disadvantage of having the term a^{-1} ($\rho^{-1}_{\epsilon/\sqrt{2}}(m)$ for us) in the bound.

For this reason, we need to prove that a is bounded away from 0, and this is the reason why Proposition B.8 is needed. On the other side, the multiplicative approach provides sharper estimates (in terms of orders of power of $|x_1 - x_2|$) for the estimation of the difference of the spatial covariances of noises $\mathcal{Z}_N$ and $\mathcal{Y}_N$ in (11), if compared to what we would get if we relied on (42). For these reasons, we chose the multiplicative approach.

Remark 3.5. The replacement of $\mathcal{Z}_N$ with $\mathcal{Y}_N$ gives a negligible error. This error is given by (11), (12), depending on the distance $|x_1 - x_2|$. We split the analysis in three cases.

- *Points* $x_1, x_2 \in D$ *such that* $|x_1 - x_2|^2 \le \epsilon^2$. Estimates (11), (12) directly imply

$$\left|\mathbb{E}\left[\mathcal{Z}_N(x_1,t)\mathcal{Z}_N(x_2,t)\right] - \mathbb{E}\left[\mathcal{Y}_N(x_1,t)\mathcal{Y}_N(x_2,t)\right]\right| \le \frac{C}{N}\cdot\frac{1}{\epsilon}\cdot\epsilon^2 \approx \mathcal{O}(\epsilon^{\theta+1}),$$
$$\left|\mathbb{E}\left[\mathcal{Z}_N(x_1,t)\mathcal{Z}_N(x_2,t)\right]\right| \le \frac{C}{N}\cdot\frac{1}{\epsilon} \approx \mathcal{O}(\epsilon^{\theta-1}).$$

- *Points* $x_1, x_2 \in D$ *such that* $|x_1 - x_2|^2 \in (\epsilon^2, \epsilon)$. Estimates (11), (12) directly imply

$$\left|\mathbb{E}\left[\mathcal{Z}_N(x_1,t)\mathcal{Z}_N(x_2,t)\right] - \mathbb{E}\left[\mathcal{Y}_N(x_1,t)\mathcal{Y}_N(x_2,t)\right]\right| \le \frac{C}{N}\cdot\frac{1}{\epsilon}\cdot\epsilon \approx \mathcal{O}(\epsilon^{\theta}),$$
$$\left|\mathbb{E}\left[\mathcal{Z}_N(x_1,t)\mathcal{Z}_N(x_2,t)\right]\right| \le \frac{C}{N}\cdot\frac{1}{\epsilon} \approx \mathcal{O}(\epsilon^{\theta-1}).$$

- *Points* $x_1, x_2 \in D$ *such that* $|x_1 - x_2|^2 \ge \epsilon$. The prefactor $N^{-1}w_{\sqrt{2\epsilon}}(x_1 - x_2)$ decays exponentially in ϵ, and both $\mathcal{Z}_N$, $\mathcal{Y}_N$ are negligible and hence interchangeable.

3.4. Nonvanishing potential $V(q)$: Modifications of proofs of main results. We show that Proposition 1.1, Proposition 1.2, and Theorem 1.3 also hold with Assumption (G) replaced by Assumption (NG).

Adaptation of the proof of Proposition 1.1 *under Assumption* (NG). In the proof of Proposition 1.1, we deal with three time-regularity estimates for the families $\{\rho_\epsilon\}_\epsilon$, $\{j_\epsilon\}_\epsilon$, $\{j_{2,\epsilon}\}_\epsilon$. In each one of them, we expand an L^p-norm of the relevant quantities (7), (8). In each case, we end up with upper bounds consisting of sums of terms labeled as ct, I_1 (and also I_2, I_3, and I_4 when applicable). If we now assume that V satisfies Assumption (NG), we can use Proposition B.6, bounds (95)–(96), to deduce the bound $\mathsf{ct} \le |t-s|^{1+\beta}$ for all three estimates. As for the remaining terms I_1 (and I_2, I_3, and I_4 when applicable), we use Proposition B.6, bounds (97)–(98), to control all terms $\mathbb{E}[\tau_1(x,s,t)]^2$ as $\mathbb{E}[\tau_1(x,s,t)]^2 \le C|t-s|$, with C independent of x and ϵ. It only remains to consider the integrals of the form

$$\begin{cases} \displaystyle\int_{\mathbb{R}} \mathbb{E}\left[(w_\epsilon(x-q(t)) - w_\epsilon(x-q(s)))^2\right]\mathrm{d}x & \text{for Step 1,} \\ \displaystyle\int_{\mathbb{R}} \mathbb{E}\left[\tau_1(x,s,t)^c\right]\mathrm{d}x, \ \ c\in\{2,3,4\} & \text{for Steps 2 and 3.} \end{cases}$$

The algebraic steps involved in the x-variable integration remain unaltered. As for the expected value of the resulting $(q(t),p(t),q(s),p(s))$-dependent quantities, the time-regularity estimates also do not change. This is a consequence of the rapidly decaying probability density function $g(t,q,p)$ and the polynomial growth of V. These facts guarantee the existence (and the correct time-dependency) of all the required moments of $q(t)-q(s)$ and $p(t)-p(s)$. As for the proofs of tightness of $\{\rho_\epsilon(\cdot,0)\}_\epsilon$, $\{j_\epsilon(\cdot,0)\}_\epsilon$, $\{j_{2,\epsilon}(\cdot,0)\}_\epsilon$, these can be adapted by using Remark B.7 for the estimates of the terms labelled ct; see, for instance, (19). ∎

Adaptation of the proof of Proposition 1.2 *under Assumption* (NG). The only change in the proof is the justification of the probability density functions of $q_1(t_j)$ and $-q_2(t_j)$, $j=1,\dots,m$, belonging to $\mathcal{S}$. This is stated in [15, Theorem 0.1]. ∎

Adaptation of the proof of Theorem 1.3 *under Assumption* (NG). The proof is identical up to, and including, estimate (38). After that, we work on (39) by using the adaptation of Proposition B.8 under Assumption (NG), whose proof is included in subsection B.3. We also need to provide estimates for the terms I_1, I_2, I_3, I_4, and ct without relying on the Gaussian setting. We define $\tilde{g}_t$ to be the probability density function of $q(t)$. We begin with ct and bound

$$\begin{aligned}\left|\mathbb{E}\left[\tau(x_1,m)\right]\right| &= \left|\int_{\mathbb{R}}\left(w_\epsilon(x_1-y)-w_\epsilon(m-y)\right)\tilde{g}_t(y)\mathrm{d}y\right|\\ &= \left|\int_{\mathbb{R}} w_\epsilon(x_1-y)\left(\tilde{g}_t(y)-\tilde{g}_t(y+m-x_1)\right)\mathrm{d}y\right|\\ &\leq \|w_\epsilon(x_1-\cdot)\|_{L^1(\mathbb{R})}\|\tilde{g}_t(\cdot)-\tilde{g}_t(\cdot+m-x_1)\|_{L^\infty(\mathbb{R})}\leq C|x_1-x_2|,\end{aligned}$$

where we have used the change of variables for q in the second equality (shift by $m-x_1$) and the boundedness of $(\partial/\partial q)g(q,p,t)$ provided by (89). This concluded the analysis of the term ct. We now turn to

$$\begin{aligned}&\mathbb{E}\left[\tau(x_1-m)^2\right]\\ &\quad=\frac{1}{\sqrt{4\pi\epsilon^2}}\Big[w_{\epsilon/\sqrt{2}}(x_1-q_1(s))+w_{\epsilon/\sqrt{2}}(m-q_1(s))\\ &\qquad-2w_{\epsilon/\sqrt{2}}\left[\frac{x_1+m}{2}-q_1(s)\right]\exp\left\{-\frac{(x_1-m)^2}{4\epsilon^2}\right\}\Big]\\ &\quad\leq\frac{1}{\sqrt{4\pi\epsilon^2}}\int_{\mathbb{R}}w_{\frac{\epsilon}{\sqrt{2}}}(x_1-y)\left(\tilde{g}_t(y)+\tilde{g}_t(y+m-x_1,t)-2\tilde{g}_t\left(y+\frac{x_1+m}{2}-x_1,t\right)\right)\mathrm{d}y\\ &\qquad+\frac{1}{\sqrt{4\pi\epsilon^2}}\frac{(x_1-m)^2}{4\epsilon^2}\int_{\mathbb{R}}w_{\frac{\epsilon}{\sqrt{2}}}\left(y-\frac{x_1+x_2}{2}\right)\tilde{g}_t(y)\mathrm{d}y\\ &\quad\leq C\left(\frac{1}{\epsilon}+\frac{1}{\epsilon^3}\right)|x_1-x_2|^2\leq\frac{C}{\epsilon^3}|x_1-x_2|^2.\end{aligned}$$

We have used (17), suitable changes of variables for q, and a second-order Taylor approximation for $\tilde{g}_t$ in the first inequality, as well as boundedness of suitable derivatives of $g(q,p,t)$ by means of (89) in the second inequality. This settles term I_3. The remaining terms I_1, I_2, and I_4 are dealt with in the same way as in the original proof. The estimation of term T_1 can be performed with the same techniques used above in the adaptation of the analysis for term T_2. ∎

3.5. Defining the regularized Dean–Kawasaki model. An immediate consequence of Theorem 1.3 is that, in a simultaneous limit of $N\to\infty$ and $\epsilon\to 0$, the stochastic noise $\mathcal{Z}_N$ in system (9) vanishes. This differs from the original Dean–Kawasaki model. However, a close approximation of such a model is recovered for a large but fixed number of particles N, by means of Theorem 1.3. We make some additional approximations to (9). These approximations are aimed at deriving a closed-expression formulation, in the variable $(\rho_\epsilon, j_\epsilon)$, for our regularized version of the Dean–Kawasaki model.

Approximation 1. We replace the noise $\mathcal{Z}_N$ with the noise $\mathcal{Y}_N$ (i.e., we neglect the remainder $\mathcal{R}_N$). This has been discussed in detail in subsections 3.2 and 3.3.

Approximation 2. With respect to the noise $\mathcal{Y}_N$, we replace $\{\rho_{\epsilon/\sqrt{2}}\}_\epsilon$ with $\{\rho_\epsilon\}_\epsilon$. This is justified by the fact that both families admit the same limit in distribution in $\mathcal{X} = C(0,T;L^2(D))$ thanks to Proposition 1.2. In addition, the noise $\mathcal{Y}_N$ features the vanishing rescaling $N^{-1/2}$, which provides an additional contribution in reducing the error caused by the replacement of $\rho_{\epsilon/\sqrt{2}}$ with ρ_ϵ.

Approximation 3. We replace the term $j_{2,\epsilon}$ with a multiple of $\frac{\partial \rho_\epsilon}{\partial x}$. This can be seen as a replacement of the random quantity $p_i^2(t)$ with its expected value. Indeed, the equilibrium state of the particle system $\{(q_i,p_i)\}_{i=1}^N$ is identified by the joint density

$$C(N,V,\sigma,\gamma)\prod_{i=1}^N \exp\left\{-\frac{2\gamma}{\sigma^2}\left(\frac{p_i^2}{2}+V(q_i)\right)\right\} = C(N,V,\sigma,\gamma)\prod_{i=1}^N M(q_i,p_i).$$

The equilibrium state shows independence between position and velocity of particles. This allows us to write

$$\mathbb{E}\big[j_{2,\epsilon}(x,t)\big] = \mathbb{E}\big[p_1^2(t)\big]\mathbb{E}\left[\frac{\partial \rho_\epsilon}{\partial x}(x,t)\right] = \frac{\sigma^2}{2\gamma}\mathbb{E}\left[\frac{\partial \rho_\epsilon}{\partial x}(x,t)\right],$$

which suggests the replacement of $j_{2,\epsilon}$ with a multiple of ρ'_ϵ. We stress the fact that at no point in this work do we assume to be working with the steady state of the particle system (2). Nevertheless, at least under Assumption (NG), the dynamics of (2) tends to the steady state for $t\to\infty$; see [15, Theorem 0.1.]. In the case $\sigma^2 \ll 2\gamma$ (i.e., for the overdamped Langevin dynamics), this entails that

$$\mathrm{Var}[p_i^2(t)] \le C\sigma^4/(2\gamma)^2 \ll \sigma^2/(2\gamma) \approx \mathbb{E}\big[p_i^2(t)\big] \approx 0.$$

It is then natural to replace p_i^2 with $\frac{\sigma^2}{2\gamma}$ on the probability space Ω, hence to replace $j_{2,\epsilon}$ with $\frac{\sigma^2}{2\gamma}\frac{\partial \rho_\epsilon}{\partial x}$.

Approximation 4. We replace the term $N^{-1}\sum_{i=1}^N V'(q_i(t))w_\epsilon(x-q_i(t))$ with the term $V'(x)\rho_\epsilon(x,t)$. This is justified by the following result, which the reader may skip on a first reading.

Lemma 3.6. *Let the scaling of N and ϵ be such that $\epsilon\to 0$ as $N\to\infty$. For each $x\in D$ and $t\in[0,T]$, we have* $\lim_{N\to\infty}[|V'(x)\rho_\epsilon(x,t) - N^{-1}\sum_{i=1}^N V'(q_i(t))\cdot w_\epsilon(x-q_i(t))|] = 0$.

Proof. The claim is trivial under Assumption (G). Let us then consider Assumption (NG). The particles being identically distributed, we only have to show that $\mathbb{E}[|V'(q_1(t)) - V'(x)|\, w_\epsilon(x-q_1(t))] \to 0$ as $\epsilon\to 0$. We use (89) to deduce that $f_q\in L^\infty(\mathbb{R})$, where f_q is the probability density function of $q_1(t)$. We set $\alpha := 2n-2\ge 0$, where n is given in Assumption (NG). In addition, we set $D_\tau(\epsilon) := [-\epsilon^{-\tau}, +\epsilon^{-\tau}]$ for some $\tau\in(0,\alpha^{-1})$ whenever $\alpha>0$, or for some $\tau>0$ when $\alpha=0$. We compute

$$\mathbb{E}\big[|V'(q_1(t))-V'(x)|\, w_\epsilon(x-q_1(t))\big] = \int_{\mathbb{R}} |V'(y)-V'(x)|\, w_\epsilon(x-y) f_q(y)\mathrm{d}y$$

$$\le C\int_{D_\tau(\epsilon)} |V'(y)-V'(x)|\, w_\epsilon(x-y)\mathrm{d}y + C\int_{D^c_\tau(\epsilon)} |V'(y)-V'(x)|\, w_\epsilon(x-y)\mathrm{d}y. \tag{43}$$

We notice that $w_\epsilon(x-y) \le C(x,\tau) w_{\tilde\epsilon}(x-y)$ for all $y \in D^c_\tau(\epsilon)$, the complement of $D_\tau(\epsilon)$, where $0 < \epsilon \le \tilde\epsilon := (|x|+1)^{-1/\tau}$. Moreover, Assumption (NG) implies that $|V'(y)| \le C(\alpha)(1+|y|^{\alpha+1})$ and $|V''(y)| \le C(\alpha)(1+|y|^\alpha)$ for all $y \in \mathbb{R}$. With respect to (43), we bound the integral on $D_\tau(\epsilon)$ by using the mean-value theorem and the control on V'', and we bound the integral on $D^c_\tau(\epsilon)$ by relying on the kernel $w_{\tilde\epsilon}$ and the control on V'. We obtain

$$\begin{aligned}
&\mathbb{E}\big[|V'(q_1(t)) - V'(x)|\, w_\epsilon(x-q_1(t))\big] \\
&\quad \le C\epsilon^{-\alpha\tau} \int_{D_\tau(\epsilon)} |y-x|\, w_\epsilon(x-y)\mathrm{d}y + C(x,\tau,\alpha)\int_{D^c_\tau(\epsilon)} (1+|y|^{\alpha+1}) w_\epsilon(x-y)\mathrm{d}y \\
&\quad \le C\epsilon^{-\alpha\tau+1} + C(x,\tau,\alpha)\int_{D^c_\tau(\epsilon)} (1+|y|^{\alpha+1}) w_{\tilde\epsilon}(x-y)\mathrm{d}y, \qquad (44)
\end{aligned}$$

where we have used Lemma A.5 in the last inequality. The right-hand side of (44) tends to 0 as $\epsilon \to 0$ due to the choice of τ and the dominated convergence theorem. This concludes the proof. □

The approximations discussed above yield the system of equations

$$\left.\begin{aligned}
&\frac{\partial \rho_\epsilon}{\partial t}(x,t) = -\frac{\partial j_\epsilon}{\partial x}(x,t), && (45a)\\
&\frac{\partial j_\epsilon}{\partial t}(x,t) = -\gamma j_\epsilon(x,t) - \left(\frac{\sigma^2}{2\gamma}\right)\frac{\partial \rho_\epsilon}{\partial x}(x,t) - V'(x)\rho_\epsilon(x,t) && (45b)\\
&\qquad\qquad + \frac{\sigma}{\sqrt{N}}\sqrt{\rho_\epsilon(x,t)}\,\tilde\xi_\epsilon, &&\\
&\rho_\epsilon(x,0) = \rho_0(x), \quad j_\epsilon(x,0) = j_0(x), &&
\end{aligned}\right\}$$

where $x \in D$, $t \in [0,T]$, and $\tilde\xi_\epsilon = Q^{1/2}_{\sqrt{2}\epsilon}\xi$ is an $L^2(D)$-valued Q-Wiener process, and ρ_0, j_0 are suitable initial conditions. System (45) is one step away from being our regularized Dean–Kawasaki model. This final step is illustrated in the final section, as the need for it shows while trying to establish existence of solutions to (45).

4. Mild solutions to the regularized Dean–Kawasaki model in a periodic setting. We investigate existence and uniqueness of mild solutions to system (14), which we refer to as a *regularized Dean–Kawasaki model*. System (14) is the 2π-periodic equivalent of (45). The reason for considering the spatially periodic case will be discussed below. Note that the quantities ρ_ϵ, j_ϵ in (45) and (14) are no longer associated with the definitions given in (7) but are the unknown solutions to the two systems.

We rewrite (45) as a stochastic partial differential equation of the type

$$\left\{\begin{aligned}
&\mathrm{d}X_\epsilon(t) = (AX_\epsilon(t) + \alpha X_\epsilon(t))\mathrm{d}t + B_N(X_\epsilon(t))\mathrm{d}W_\epsilon,\\
&X_\epsilon(0) = X_0,
\end{aligned}\right. \qquad (46)$$

where $X_\epsilon(t) := (\rho_\epsilon(\cdot,t), j_\epsilon(\cdot,t))$, $X_0 = (\rho_0, j_0)$, and $W_\epsilon := (W_{\epsilon,1}, W_{\epsilon,2})$ is a suitable stochastic noise, with

$$AX_\epsilon(t) := \left(-\frac{\partial j_\epsilon}{\partial x}(\cdot,t),\, -\gamma j_\epsilon(\cdot,t) - \left(\frac{\sigma^2}{2\gamma}\right)\frac{\partial\rho_\epsilon}{\partial x}(\cdot,t)\right), \quad \alpha X_\epsilon(t) := (0, -V'(\cdot)\rho_\epsilon(\cdot,t)),$$

and B_N is some suitable integrand specified below.

Subsection 4.1 is devoted to the analysis of the operator A by means of the C_0-semigroup theory. We define and analyze the periodic equivalents $W_{\mathrm{per},\epsilon}$ and α_{per}

of W_ϵ and α in subsection 4.2. We describe the relevant properties of the stochastic integrand B_N in subsection 4.3 and prove existence and uniqueness of mild solutions to a suitable locally Lipschitz approximation of (14) in subsection 4.4. We then prove suitable small-noise regime estimates in subsections 4.5 and 4.6. We finally prove the main existence and uniqueness result, Theorem 1.6, in subsection 4.7.

In this section, we set $D := [0, 2\pi]$. We fix $k_B T_e = \sigma^2/(2\gamma) := 1$ for notational simplicity, even though all our conclusions hold for arbitrary positive ratio $\sigma^2/(2\gamma)$.

4.1. Semigroup analysis for the operator $\boldsymbol{A}$ in $\boldsymbol{\mathcal{W} = H^1_{\mathrm{per}}(D) \times H^1_{\mathrm{per}}(D)}$. We characterize the semigroup associated with the operator A, which can be done in a straightforward manner. For any 2π-periodic function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f|_D \in L^2(D)$, we write its Fourier coefficients as $\hat{f}_m := (2\pi)^{-1} \int_D e^{-imx} f(x)\mathrm{d}x$ for any $m \in \mathbb{Z}$. We consider the Sobolev spaces of 2π-periodic functions

$$H^n_{\mathrm{per}}(D) := \left\{ f = \sum_{m\in\mathbb{Z}} \hat{f}_m e^{-imx} \colon \sum_{m\in\mathbb{Z}} \left(1+m^2\right)^n \hat{f}_m^2 < \infty \right\}, \quad n \in \mathbb{N},$$

endowed with standard norms and inner products. We also consider the spaces

$$C^n_{\mathrm{per}}(D) := \left\{ f : f \in C^n(\mathbb{R}), f \text{ is periodic with period } 2\pi \right\}, \quad n \in \mathbb{N} \cup \{0\},$$

where $C^0_{\mathrm{per}}(D)$ is endowed with its standard norm. We also recall the following Sobolev embedding theorem, valid only in one space dimension.

PROPOSITION 4.1. *The embedding $H^1_{\mathrm{per}}(D) \subset C^0_{\mathrm{per}}(D)$ is continuous.*

As an immediate consequence of Proposition 4.1, we deduce that, for $f \in H^n_{\mathrm{per}}(D)$, $n \geq 1$,

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} f(0) = \frac{\mathrm{d}^k}{\mathrm{d}x^k} f(2\pi) \qquad \text{for all } k = 0, 1, \ldots, n-1.$$

We also recall the spaces

$$\begin{aligned}
&\mathcal{W} := H^1_{\mathrm{per}}(D) \times H^1_{\mathrm{per}}(D), \\
&\quad \langle (u_1, v_1), (u_2, v_2) \rangle_{\mathcal{W}} := \langle u_1, u_2 \rangle_{H^1_{\mathrm{per}}(D)} + \langle v_1, v_2 \rangle_{H^1_{\mathrm{per}}(D)}, \\
&\mathcal{W} \supset \mathcal{D}(A) := H^2_{\mathrm{per}}(D) \times H^2_{\mathrm{per}}(D), \\
&\quad \langle (u_1, v_1), (u_2, v_2) \rangle_{\mathcal{D}(A)} := \langle u_1, u_2 \rangle_{H^2_{\mathrm{per}}(D)} + \langle v_1, v_2 \rangle_{H^2_{\mathrm{per}}(D)}.
\end{aligned}$$

LEMMA 4.2. *The operator $A\colon \mathcal{D}(A) \subset \mathcal{W} \to \mathcal{W}$ defines a C_0-semigroup of contractions $\{S(t)\}_{t\geq 0}$.*

Proof. We verify the assumptions of the Hille–Yosida theorem, as stated in [28, Theorem 3.1]. This is a straightforward step and might be skipped on a first reading.

A is a closed operator, and $\mathcal{D}(A)$ is dense in $\mathcal{W}$. This is easily checked.

The resolvent set of A contains the positive half line. For every $\lambda > 0$, we consider $A_\lambda^{-1} := (A - \lambda I)^{-1}$ whenever this is well-defined. We first prove that it exists, by showing injectivity of $A_\lambda := A - \lambda I$. Let us then assume that $A_\lambda(\rho, j) = (0, 0)$. We multiply the first component of $A_\lambda(\rho, j)$ by ρ and the second component of $A_\lambda(\rho, j)$ by j, and we obtain

$$(-j' - \lambda\rho)\rho + (-(\lambda+\gamma)j - \rho')j = -\lambda\rho^2 - (\lambda+\gamma)j^2 - (\rho j)' = 0.$$

Integrating over D and using the periodic boundary conditions for ρ and j, we obtain

$$\lambda \|\rho\|^2_{L^2(D)} + (\lambda+\gamma)\|j\|^2_{L^2(D)} = 0.$$

Since $\lambda, \gamma > 0$, we deduce that $(\rho, j) = (0, 0)$. We now show that A_λ^{-1} is a bounded operator. Consider $A_\lambda^{-1}(a, b) = (\rho, j)$. This implies

$$\lambda\rho = -a - j', \tag{47}$$

$$(\lambda+\gamma)j = -b - \rho', \tag{48}$$

$$\lambda\rho' = -a' - j'', \tag{49}$$

$$(\lambda+\gamma)j' = -b' - \rho'', \tag{50}$$

where (49) (respectively, (50)) is obtained by differentiating (47) (respectively, (48)). We multiply (47) by ρ, (48) by j, (49) by ρ', and (50) by j' and sum the four equalities. An integration of the resulting expression over D yields

$$\begin{aligned}\lambda\|(\rho, j)\|^2_{\mathcal{W}} &\leq \lambda\|\rho\|^2_{H^1_{\mathrm{per}}(D)} + (\lambda+\gamma)\|j\|^2_{H^1_{\mathrm{per}}(D)}\\ &= \int_D -a\rho\,\mathrm{d}x + \int_D -bj\,\mathrm{d}x + \int_D -a'\rho'\,\mathrm{d}x + \int_D -b'j'\,\mathrm{d}x,\end{aligned} \tag{51}$$

where we have also used the periodic boundary conditions for ρ, j, ρ', j'. We now use the Cauchy–Schwarz inequality and the Young inequality $|xy| \leq \theta^2x^2 + (1/4\theta^2)y^2$ with $\theta^2 := \lambda/2$ to bound the four integrals in the right-hand side of (51). This directly gives $(\lambda/2)\|(\rho, j)\|^2_{\mathcal{W}} \leq (1/2\lambda)\|(a, b)\|^2_{\mathcal{W}}$, which implies

$$\|A_\lambda^{-1}\|_{\mathcal{L}(\mathcal{W},\mathcal{W})} \leq \frac{1}{\lambda}, \tag{52}$$

so A_λ^{-1} is bounded. We now show that $\mathrm{Dom}(A_\lambda^{-1})$ is dense in $\mathcal{W}$. Let us fix $(a, b) \in H^2_{\mathrm{per}}(D) \times H^1_{\mathrm{per}}(D)$. We consider the system of equations $A_\lambda(\rho, j) = (a, b)$, namely,

$$-j' - \lambda\rho = a, \qquad -(\lambda+\gamma)j - \rho' = b.$$

We rewrite the first equation as $\rho = (-j' - a)/\lambda$ and substitute into the second equation, obtaining

$$-\frac{j''}{\lambda} + (\lambda+\gamma)j = \frac{a'}{\lambda} - b \in H^1_{\mathrm{per}}(D). \tag{53}$$

The elliptic theory provides existence of a unique solution $j \in H^3_{\mathrm{per}}(D)$ for (53). From $\rho := (-j' - a)/\lambda$, we immediately deduce that $\rho \in H^2_{\mathrm{per}}(D)$. We have shown that, for every (a, b) in a dense subset of $\mathcal{W}$ (namely, $H^2_{\mathrm{per}}(D) \times H^1_{\mathrm{per}}(D)$), the operator A_λ^{-1} is well-defined.

Inequality [28, (3.1)] *is satisfied.* This is precisely (52). □

4.2. Introducing periodic noise and periodic potential drift. We now define the noise W_ϵ for (46) in accordance with the noise in (45b). We set

$$\dot{W}_\epsilon := \left(0,\, \tilde{\xi}_\epsilon\right) = \left(0,\, Q^{1/2}_{\sqrt{2}\epsilon}\xi\right).$$

The second component of $\dot{W}_\epsilon$ agrees with the noise in (45b). Since (45a) is a deterministic equation, we set the first component of $\dot{W}_\epsilon$ to zero. We represent W_ϵ as [29, Proposition 2.1.10]

$$W_\epsilon = \sum_{j=1}^{\infty} \sqrt{\lambda_j}(0, e_j)\beta_j(t), \tag{54}$$

where $\{e_j\}_j$ and $\{\lambda_j\}_j$ refer to the families of eigenfunctions and eigenvalues of the Hilbert–Schmidt integral operator $Q_{\sqrt{2}\epsilon}$ on $L^2(D)$. Unfortunately, the eigenfunctions $\{e_j\}_j$ are not 2π-periodic. To verify this, one can rely on Mercer's theorem and evaluate the kernel expansion $w_{\sqrt{2}\epsilon}(x-y) = \sum_{j=1}^{\infty} \lambda_j e_j(x)e_j(y)$ for the pairs $(x,y) = (0,0)$ and $(x,y) = (0,2\pi)$. We deduce that the Q-Wiener process W_ϵ does *not* necessarily take values in the space associated with the semigroup analysis of A, i.e., in $\mathcal{W}$. In order to resolve this issue, we identify the end-points of the interval $[0,2\pi]$, thus thinking of $[0,2\pi]$ as a flat torus. We provide, for each $\epsilon > 0$, a 2π-periodic kernel $p_{\sqrt{2}\epsilon}$ approximating $w_{\sqrt{2}\epsilon}$. A suitable choice lies in the von Mises distribution, a 2π-periodic distribution parametrized by $\mu \in \mathbb{R}$, $\kappa > 0$, and given by the probability density function

$$f(x,\mu,\kappa) = \frac{e^{\kappa\cos(x-\mu)}}{2\pi I_0(\kappa)}, \qquad I_0(\kappa) := \frac{1}{2\pi}\int_D e^{\kappa\cos(x)}\mathrm{d}x.$$

The von Mises distribution [14] approximates the Gaussian kernel in the following way:

$$\lim_{\kappa\to+\infty} \left\| f(x,\mu,\kappa) - \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}\right\|_{C^0(\mu-\pi,\,\mu+\pi)} = 0, \quad \text{where } \sigma^2 := \kappa^{-1}.$$

For this reason, we replace the kernel $w_{\sqrt{2}\epsilon}$, $\epsilon > 0$, with the 2π-periodic kernel

$$p_{\sqrt{2}\epsilon}(x) := f\big(x,0,(2\epsilon^2)^{-1}\big) = \frac{e^{\frac{\cos(x)}{2\epsilon^2}}}{2\pi I_0(1/(2\epsilon^2))} = Z^{-1}_{\sqrt{2}\epsilon} e^{-\frac{\sin^2(x/2)}{\epsilon^2}}, \quad Z^{-1}_{\sqrt{2}\epsilon} := \frac{e^{\frac{1}{2\epsilon^2}}}{2\pi I_0(1/(2\epsilon^2))}.$$

In the limit $\epsilon \to 0$, the kernel $p_{\sqrt{2}\epsilon}$ recovers the Gaussian kernel $w_{\sqrt{2}\epsilon}$ on the flat torus. We study the eigenfunctions and eigenvalues of the operator

$$(55) \quad P_{\sqrt{2}\epsilon}\colon L^2(D) \to L^2(D), \qquad P_{\sqrt{2}\epsilon} f(x) = \int_D p_{\sqrt{2}\epsilon}(x-y)f(y)\mathrm{d}y, \quad f \in L^2(D).$$

We obtain the eigenfunctions $\{e_{j,\epsilon}\}_{j\in\mathbb{Z}}$ and eigenvalues $\{\lambda_{j,\epsilon}\}_{j\in\mathbb{Z}}$ of $P_{\sqrt{2}\epsilon}$ from [10, section 4.2], namely,

$$e_{j,\epsilon}(x) = e_j(x) = \begin{cases} \sqrt{\frac{1}{\pi}}\cos(jx) & \text{if } j > 0, \\ \sqrt{\frac{1}{\pi}}\sin(jx) & \text{if } j < 0, \\ \sqrt{\frac{1}{2\pi}} & \text{if } j = 0, \end{cases}$$

and

$$(56) \quad \lambda_{j,\epsilon} = \begin{cases} Z^{-1}_{\sqrt{2}\epsilon}\displaystyle\int_D e^{-\frac{\sin^2(x/2)}{\epsilon^2}}\cos(jx)\mathrm{d}x = C_2 Z^{-1}_{\sqrt{2}\epsilon} e^{-\frac{1}{2\epsilon^2}} I_j\left(\{2\epsilon^2\}^{-1}\right) & \text{if } j \neq 0, \\ 1 & \text{if } j = 0, \end{cases}$$

where $I_j(z) := (2\pi)^{-1}\int_D e^{z\cos(x)}\cos(jx)\mathrm{d}x$ is the *modified Bessel function* of first kind and order j; see [1, equation (9.6.19)]. It is immediate to notice that $\{e_j\}_j$ is an orthogonal basis of $H^1_{\mathrm{per}}(D)$, and that the family $\{f_j\}_{j\in\mathbb{Z}}$

$$(57) \qquad f_j(x) = \begin{cases} e_j(x)/\sqrt{1+j^2} & \text{if } j \neq 0, \\ \sqrt{\frac{1}{2\pi}} & \text{if } j = 0, \end{cases}$$

is an orthonormal basis of $H^1_{\mathrm{per}}(D)$. This is crucial, as it will allow us to construct a $\mathcal{W}$-valued noise below.

We now turn to estimating relevant properties of $\{\lambda_{j,\epsilon}\}_j$.

LEMMA 4.3. *Fix $n \in \mathbb{N}$. There exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ we have $\sum_{j\in\mathbb{Z}} \lambda_{j,\epsilon}|j|^n \leq C(n)\epsilon^{-(2n+3)}$.*

Proof. We start with bounding $Z_{\sqrt{2}\epsilon}$ from below as

$$Z_{\sqrt{2}\epsilon} = \int_D e^{-\frac{\sin^2(x/2)}{\epsilon^2}}\mathrm{d}x \geq \int_D e^{-\frac{x^2}{4\epsilon^2}}\mathrm{d}x \geq \int_0^{\sqrt{4\epsilon^2\ln 2}} (1/2)\mathrm{d}x = C\epsilon. \tag{58}$$

We now turn to I_j. We first of all notice that $I_1(z) \leq I_0(z)$ for any $z \geq 0$. In addition, we have

$$I_0(z) = (2\pi)^{-1}\int_D e^{z\cos(x)}\mathrm{d}x \leq \int_D e^z\mathrm{d}x = Ce^z.$$

We use a recursive property of the modified Bessel functions of first kind [1, equation (9.6.26)], namely,

$$I_{j+1}(z) = I_{j-1}(z) - \frac{2j}{z}I_j(z) \qquad \text{for all } z > 0, \text{ for all } j \in \mathbb{N}. \tag{59}$$

Since the modified Bessel functions of first kind are always nonnegative for nonnegative arguments [1, equation (9.6.10)], we deduce from (59) that $I_j(z) \leq (z/2j)I_{j-1}(z)$. For $j > z$, we have $I_j(z) \leq (1/2)I_{j-1}(z)$, which implies an exponential decay of $I_j(z)$ for $j > z$. Since $I_1(z) \leq I_0(z)$, equality (59) also implies that $I_j(z) \leq I_0(z)$ for all $j \in \mathbb{N}$. To sum up, we get the bounds

$$I_j(z) \leq \begin{cases} Ce^z & \text{if } j \leq z, \\ Ce^z\left(\frac{1}{2}\right)^{j-z} & \text{if } j > z. \end{cases} \tag{60}$$

We take $z = (2\epsilon^2)^{-1}$, and we set $m(\epsilon) := \lceil (2\epsilon^2)^{-1}\rceil$. We feed (58) and (60) into (56), thus obtaining

$$\lambda_{j,\epsilon} \leq \begin{cases} C\epsilon^{-1} & \text{if } j \leq m(\epsilon), \\ C\epsilon^{-1}\left(\frac{1}{2}\right)^{j-m(\epsilon)} & \text{if } j > m(\epsilon), \end{cases} \tag{61}$$

where C is a constant independent of ϵ. As a result of (61) we get, for ϵ sufficiently small,

$$\begin{aligned}\frac{1}{2}\sum_{j\in\mathbb{Z}}\lambda_{j,\epsilon}|j|^n \leq \sum_{j=0}^{\infty}\lambda_{j,\epsilon}j^n = \sum_{j=0}^{m(\epsilon)}\lambda_{j,\epsilon}j^n + \sum_{j>m(\epsilon)}\lambda_{j,\epsilon}j^n &\leq C(n)\epsilon^{-1}m(\epsilon)^{(n+1)} \\ &\quad + C(n)\epsilon^{-1}\sum_{j>m(\epsilon)}(1/2)^{j-m(\epsilon)}\{(j-m(\epsilon))^n + m(\epsilon)^n\} \\ &\leq C(n)\epsilon^{-(2n+3)},\end{aligned}$$

and the proof is complete. ∎

These considerations show that the noise $\dot{W}_\epsilon$ given in (54) can be replaced, in a periodic setting, by the noise $\dot{W}_{\mathrm{per},\epsilon} = (0, \tilde{\xi}_{\mathrm{per},\epsilon}) := (0, P^{1/2}_{\sqrt{2}\epsilon}\xi)$, where P is defined in (55). This noise is a $\mathcal{W}$-valued Q-Wiener process given by

$$W_{\mathrm{per},\epsilon} = \sum_{j\in\mathbb{Z}}\sqrt{\alpha_{j,\epsilon}}(0, f_j)\beta_j, \qquad \alpha_{j,\epsilon} := (1+j^2)\lambda_{j,\epsilon}, \tag{62}$$

where $\{\beta_j\}_j$ is a family of independent one-dimensional standard Brownian motions. For consistency, we assume V is periodic, i.e., $V = V_{\mathrm{per}} \in C^2_{\mathrm{per}}(D)$. It is also immediate to notice that the operator $\alpha_{\mathrm{per}} X_\epsilon(t) := \left(0, -V'_{\mathrm{per}}(\cdot)\rho_\epsilon(\cdot,t)\right)$ belongs to $L(\mathcal{W})$, i.e., to the set of bounded linear operators on $\mathcal{W}$.

In the remainder of the paper, we investigate existence and uniqueness of solutions to the *regularized Dean–Kawasaki model*

$$\left\{ \begin{array}{l} \mathrm{d}X_\epsilon(t) = (AX_\epsilon(t) + \alpha_{\mathrm{per}} X_\epsilon(t))\mathrm{d}t + B_N(X_\epsilon(t))\mathrm{d}W_{\mathrm{per},\epsilon}, \\ X_\epsilon(0) = X_0. \end{array} \right. \tag{63}$$

System (63) is the equivalent of (45) in a periodic setting and is a functional rewriting of (14).

4.3. Locally Lipschitz stochastic integrand with respect to $\mathcal{W}$-topology. In this subsection, we define and analyze the properties of the noise integrand B_N. It is natural to define $B_N \colon \mathcal{W} \to \{f \colon \mathcal{W} \to L^2(D) \times L^2(D)\}$ as

$$B_N((\rho,j))(a,b) := \frac{\sigma}{\sqrt{N}}\left(0,\ \sqrt{|\rho|}\cdot b\right).$$

Remark 4.4. We see that

$$\int_0^t B_N((X(s),Y(s)))\mathrm{d}W_{\mathrm{per},\epsilon}(s) = \int_0^t \sum_{j\in\mathbb{Z}} \sqrt{\alpha_{j,\epsilon}} B_N((X(s),Y(s)))(0,f_j)\mathrm{d}\beta_j(s)$$

$$= \frac{\sigma}{\sqrt{N}} \int_0^t \sum_{j\in\mathbb{Z}} \sqrt{\alpha_{j,\epsilon}}\left(0, \sqrt{|X(s)|}f_j\right)\mathrm{d}\beta_j(s) = \left(0, \int_0^t \frac{\sigma}{\sqrt{N}}\sqrt{|X(s)|}\mathrm{d}P^{1/2}_{\sqrt{2}\epsilon}\xi(s)\right). \tag{64}$$

The last expression of (64) is precisely the stochastic noise of (63).

The integrand B_N poses several difficulties. First, B_N is not a mapping from $\mathcal{W}$ to $L^0_2(\mathcal{W})$, where $L^0_2(\mathcal{W})$ denotes the set of Hilbert–Schmidt operators from $P^{1/2}_{\sqrt{2}\epsilon}\mathcal{W} \subset \mathcal{W}$ into $\mathcal{W}$; see [29, section 2.3]. Second, B_N is not Lipschitz or locally Lipschitz with respect to (ρ, j). Both problems are due to the singularity of the square-root function. We address both problems by regularizing this singularity. For some $\delta > 0$, we define

$$B_{N,\delta}((\rho,j))(a,b) := \frac{\sigma}{\sqrt{N}}\left(0,\ h_\delta(\rho)\cdot b\right),$$

where $h_\delta \colon \mathbb{R} \to \mathbb{R}$ is a C^2-Lipschitz modification of $\sqrt{|z|}$ in $[-\delta, +\delta]$. In this way, h_δ is Lipschitz and has bounded first and second derivatives. We characterize some important features of $B_{N,\delta}$.

Lemma 4.5. *The following properties hold:*
(i) *$B_{N,\delta}$ is a map from $\mathcal{W}$ to $L(\mathcal{W})$.*
(ii) *$B_{N,\delta}$ is locally Lipschitz with respect to the $L^0_2(\mathcal{W})$-norm.*
(iii) *$B_{N,\delta}$ has sublinear growth at infinity with the respect to the $L^0_2(\mathcal{W})$-norm.*

Proof. Statement (i). Take $(u,v),(a,b) \in \mathcal{W}$. We use Proposition 4.1 and write

$$\begin{aligned}\|B_{N,\delta}((u,v))(a,b)\|^2_{\mathcal{W}} = \frac{\sigma^2}{N}\|h_\delta(u)b\|^2_{H^1_{\mathrm{per}}(D)} &\le \frac{\sigma^2}{N}\Big\{\|h_\delta(u)b\|^2_{L^2(D)} + C(\delta,u)\|b'\|^2_{L^2(D)} \\ &\quad + C(\delta)\|b\|^2_{C^0_{\mathrm{per}}(D)}\|u'\|^2_{L^2(D)}\Big\} \\ &\le \frac{\sigma^2}{N}C(\delta,u)\|b\|^2_{H^1_{\mathrm{per}}(D)} \le \frac{\sigma^2}{N}C(\delta,u)\|(a,b)\|^2_{\mathcal{W}}.\end{aligned}$$

This settles the first claim.

Statement (ii). Take $(u_1, v_1), (u_2, v_2) \in \mathcal{W}$, such that $\|(u_1, v_1)\|_{\mathcal{W}} \leq k$, $\|(u_2, v_2)\|_{\mathcal{W}} \leq k$. We have

$$\begin{aligned}
&\|B_{N,\delta}((u_1, v_1)) - B_{N,\delta}((u_2, v_2))\|^2_{L_2^0(\mathcal{W})} \\
&\quad = \sum_{j\in\mathbb{Z}} \|\sqrt{\alpha_{j,\epsilon}}\,\{B_{N,\delta}((u_1, v_1)) - B_{N,\delta}((u_2, v_2))\}\,(0, f_j)\|^2_{\mathcal{W}} \\
&\quad = \frac{\sigma^2}{N}\sum_{j\in\mathbb{Z}} \alpha_{j,\epsilon}\,\|(0, \{h_\delta(u_1) - h_\delta(u_2)\}\, f_j)\|^2_{\mathcal{W}}.
\end{aligned}$$

The right-hand side in the expression above is well-defined by (i). From (57), we deduce that $\|f_j\|_{L^\infty} \leq \pi^{-1/2}$, $\|f_j'\|_{L^\infty} \leq \pi^{-1/2}$, for all $j \in \mathbb{Z}$. We use this fact, as well as the boundedness of h_δ', to compute

$$\begin{aligned}
&\frac{\sigma^2}{N}\sum_{j\in\mathbb{Z}} \alpha_{j,\epsilon}\,\|(0, \{h_\delta(u_1) - h_\delta(u_2)\}\, f_j)\|^2_{\mathcal{W}} \\
&\quad \leq \frac{\sigma^2}{N}\left[\sum_{j\in\mathbb{Z}} \alpha_{j,\epsilon}\,\|\{h_\delta(u_1) - h_\delta(u_2)\}\, f_j\|^2_{L^2(D)}\right. \\
&\qquad \left. + \sum_{j\in\mathbb{Z}} \alpha_{j,\epsilon}\left\|\frac{\mathrm{d}}{\mathrm{d}x}(\{h_\delta(u_1) - h_\delta(u_2)\}\, f_j)\right\|^2_{L^2(D)}\right] \\
&\quad \leq C\frac{\sigma^2}{N}\left[\sum_{j\in\mathbb{Z}} \alpha_{j,\epsilon}\,\|h_\delta(u_1) - h_\delta(u_2)\|^2_{L^2(D)} + \sum_{j\in\mathbb{Z}} \alpha_{j,\epsilon}\left\|\frac{\mathrm{d}}{\mathrm{d}x}\,\{h_\delta(u_1) - h_\delta(u_2)\}\right\|^2_{L^2(D)}\right] \\
&\quad \leq C(\delta)\frac{\sigma^2}{N}\left(\sum_{j\in\mathbb{Z}} \alpha_{j,\epsilon}\right)\Big\{\|u_1 - u_2\|^2_{L^2(D)} + \|h_\delta'(u_1)(u_1' - u_2')\|^2_{L^2(D)} \\
&\qquad + \|u_2'(h_\delta'(u_1) - h_\delta'(u_2))\|^2_{L^2(D)}\Big\}.
\end{aligned}$$

We use Proposition 4.1, the boundedness of h_δ', h_δ'', and Lemma 4.3 to deduce

$$\begin{aligned}
&\|B_{N,\delta}((u_1, v_1)) - B_{N,\delta}((u_2, v_2))\|^2_{L_2^0(\mathcal{W})} \\
&\quad \leq C(\delta)\frac{\sigma^2}{N}\left(\sum_{j\in\mathbb{Z}} \alpha_{j,\epsilon}\right)\Big\{\|u_1 - u_2\|^2_{L^2(D)} + \|u_1' - u_2'\|^2_{L^2(D)} \\
&\qquad + \|u_2'\|^2_{L^2(D)}\|u_1 - u_2\|^2_{C^0_{\mathrm{per}}(D)}\Big\} \leq C(\delta, k)\frac{\sigma^2}{N}\epsilon^{-7}\|u_1 - u_2\|^2_{H^1_{\mathrm{per}}(D)} \\
&\quad \leq C(\delta, k)\frac{\sigma^2}{N}\epsilon^{-7}\|(u_1, v_1) - (u_2, v_2)\|^2_{\mathcal{W}},
\end{aligned}$$

which is the desired local Lipschitz property for $B_{N,\delta}$.

Statement (iii). We proceed similarly to the proof of (ii) and compute

$$\begin{aligned}
\|B_{N,\delta}((u, v))\|^2_{L_2^0(\mathcal{W})} &= \sum_{j\in\mathbb{Z}} \|\sqrt{\alpha_{j,\epsilon}} B_{N,\delta}((u, v))(0, f_j)\|^2_{\mathcal{W}} = \frac{\sigma^2}{N}\sum_{j\in\mathbb{Z}} \alpha_{j,\epsilon}\,\|(0, h_\delta(u) f_j)\|^2_{\mathcal{W}} \\
&\leq \frac{\sigma^2}{N}\left[\sum_{j\in\mathbb{Z}} \alpha_{j,\epsilon}\,\|h_\delta(u) f_j\|^2_{L^2(D)} + \sum_{j\in\mathbb{Z}} \alpha_{j,\epsilon}\left\|\frac{\mathrm{d}}{\mathrm{d}x}(h_\delta(u) f_j)\right\|^2_{L^2(D)}\right]
\end{aligned}$$

$$\le C\frac{\sigma^2}{N}\left[\sum_{j\in\mathbb{Z}}\alpha_{j,\epsilon}\,\|h_\delta(u)\|^2_{L^2(D)}+\sum_{j\in\mathbb{Z}}\alpha_{j,\epsilon}\|h'_\delta(u)u'\|^2_{L^2(D)}\right]$$
$$\le C(\delta)\frac{\sigma^2}{N}\left[\sum_{j\in\mathbb{Z}}\alpha_{j,\epsilon}\right]\left(1+\|(u,v)\|^2_{\mathcal{W}}\right)=C(\delta)\frac{\sigma^2}{N}\epsilon^{-7}\left(1+\|(u,v)\|^2_{\mathcal{W}}\right),$$

where the last inequality follows from the sublinearity of h_δ at infinity and the boundedness of h'_δ. We deduce

(65)
$$\|B_\delta((u,v))\|_{L^0_2(\mathcal{W})}\le\sqrt{C(\delta)\frac{\sigma^2}{N}\epsilon^{-7}}\,(1+\|(u,v)\|_{\mathcal{W}})=C(\delta)\sigma\underbrace{N^{-1/2}\epsilon^{-7/2}}_{=:M(\epsilon,N)}(1+\|(u,v)\|_{\mathcal{W}}).$$

This completes the proof. ∎

Remark 4.6. The quantity $M(\epsilon,N)$ introduced in (65) is the justification of the scaling $\theta>7$ in Theorem 1.6.

4.4. Existence of mild solutions in the $\mathcal{W}$-topology up to random time. We consider the following δ-smoothed version of the regularized Dean–Kawasaki system (63):

$$\left\{\begin{array}{l}\mathrm{d}X_{\epsilon,\delta}(t)=(AX_{\epsilon,\delta}(t)+\alpha_{\mathrm{per}}X_{\epsilon,\delta}(t))\mathrm{d}t+B_{N,\delta}(X_{\epsilon,\delta}(t))\mathrm{d}W_{\mathrm{per},\epsilon},\\ X_{\epsilon,\delta}(0)=X_0.\end{array}\right. \tag{66}$$

We prove the following result.

PROPOSITION 4.7. *Let $T>0$. Let $X_0\in\mathcal{W}$ be deterministic. Then* (66) *admits a unique mild solution $X_{\epsilon,\delta}$ on $[0,T]$ with respect to the $\mathcal{W}$-topology. Moreover, the solution $X_{\epsilon,\delta}$ is càdlàg in the $\mathcal{W}$-topology.*

Let $\{S(t)\}_{t\ge0}$ be the C_0-semigroup generated by A discussed in Lemma 4.2. We recall that a *mild solution* for (66) is [7, Chapter 7] a predictable $\mathcal{W}$-valued process $X_{\epsilon,\delta}(t)=(\rho_{\epsilon,\delta}(t),j_{\epsilon,\delta}(t))$, $t\in[0,T]$, such that

$$\mathbb{P}\left(\int_0^T\|X_{\epsilon,\delta}(s)\|^2_{\mathcal{W}}\,\mathrm{d}s<\infty\right)=1, \tag{67}$$

and, for arbitrary $t\in[0,T]$,

$$X_{\epsilon,\delta}(t)=S(t)X_0+\int_0^t S(t-s)\alpha_{\mathrm{per}}X_{\epsilon,\delta}(s)\mathrm{d}s+\int_0^t S(t-s)B_{N,\delta}(X_{\epsilon,\delta}(s))\mathrm{d}W_{\mathrm{per},\epsilon},\quad \mathbb{P}\text{-a.s.}$$

Proof of Proposition 4.7. We apply [31, Theorem 4.5] and take into account [31, Remark 4.6]. ∎

The mild solution $X_{\epsilon,\delta}$ to (66) is, in particular, càdlàg at time $t=0$ with respect to the $\mathcal{W}$-norm. Let us fix a parameter $\eta>\delta>0$. In addition to the hypotheses already given for X_0 in Proposition 4.7, we also assume

$$\rho_0(x)\ge\eta\qquad\text{for all } x\in D. \tag{68}$$

Keeping in mind Proposition 4.1 and the càdlàg properties at time $t=0$, we deduce the existence of a random time $\zeta(\omega)$ such that

$$\|\rho_0(\cdot)-\rho(t,\cdot)\|_{L^\infty(D)}\le\eta-\delta\qquad\text{for all } t\in[0,\zeta(\omega)). \tag{69}$$

The bound (69) implies that $B_{N,\delta}(X_{\epsilon,\delta}(s))$ coincides with $B_N(X_{\epsilon,\delta}(s))$ for $s\in[0,\zeta(\omega))$. We thus have the following.

THEOREM 4.8. *Let the hypotheses of Proposition* 4.7 *be satisfied, as well as* (68). *Then the regularized Dean–Kawasaki model* (63) *admits a unique mild solution with respect to the $\mathcal{W}$-topology up to a random time ζ.*

4.5. Estimates for $\boldsymbol{X_{\epsilon,\delta}}$. We now study some moment bounds for the real-valued random variables $\|X_{\epsilon,\delta}(t)\|_{\mathcal{W}}$, where $X_{\epsilon,\delta}$ solves (66).

PROPOSITION 4.9. *Let $T>0$, $\delta>0$, and $q>2$ be fixed. Let $X_0\in\mathcal{W}$ be a deterministic initial condition for* (66). *Let $\Theta=\Theta(T,q,\sigma,\delta,\epsilon,N):=\{C(q,T)\|X_0\|^q_{\mathcal{W}} + TC(\sigma,\delta)M^q(\epsilon,N)\}e^{C(T,q)+C(T,\sigma,\delta)M^q(\epsilon,N)}$. Then*

$$\sup_{t\in[0,T]}\mathbb{E}\big[\|X_{\epsilon,\delta}(t)\|^q_{\mathcal{W}}\big]\le\Theta. \tag{70}$$

Proof. We rely on some ideas of the proof of [7, Theorem 7.2]. We know from Proposition 4.7 that the paths of $X_{\epsilon,\delta}$ are càdlàg in the $\mathcal{W}$-topology. It follows that the real-valued process $t\mapsto\|X_{\epsilon,\delta}(t)\|^q_{\mathcal{W}}$ is also càdlàg. This fact, together with (65), allows us to deduce

$$\int_0^T\|B_{N,\delta}(X_{\epsilon,\delta}(s))\|^q_{L_2^0(\mathcal{W})}\,\mathrm{d}s<\infty,\qquad\int_0^T\|\alpha_{\mathrm{per}}(X_{\epsilon,\delta}(s))\|_{\mathcal{W}}\,\mathrm{d}s<\infty,\qquad\mathbb{P}\text{-a.s.} \tag{71}$$

For $R\in\mathbb{N}$, we define the stopping times

$$\begin{aligned}\tau_R:=\inf\Bigg\{t\in(0,T]:&\int_0^t\|B_{N,\delta}(X_{\epsilon,\delta}(s))\|^q_{L_2^0(\mathcal{W})}\,\mathrm{d}s\ge R\\ \text{or}\quad&\int_0^t\|\alpha_{\mathrm{per}}(X_{\epsilon,\delta}(s))\|_{\mathcal{W}}\,\mathrm{d}s\ge R\Bigg\},\end{aligned}$$

with the usual convention $\tau_R:=T$ whenever the above infimum acts on the empty set. If we set $X_{\epsilon,\delta,R}(t):=\mathbf{1}_{[0,\tau_R]}(t)X_{\epsilon,\delta}(t)$, it is then clear that

$$\begin{aligned}X_{\epsilon,\delta,R}(t)=\mathbf{1}_{[0,\tau_R]}(t)S(t)X_0+\mathbf{1}_{[0,\tau_R]}(t)\int_0^t\mathbf{1}_{[0,\tau_R]}(s)S(t-s)\alpha_{\mathrm{per}}X_{\epsilon,\delta,R}(s)\mathrm{d}s\\ +\mathbf{1}_{[0,\tau_R]}(t)\int_0^t\mathbf{1}_{[0,\tau_R]}(s)S(t-s)B_{N,\delta}(X_{\epsilon,\delta,R}(s))\mathrm{d}W_{\mathrm{per},\epsilon}.\end{aligned}$$

We rely on [7, Theorem 4.36], (65), and the Hölder inequality and deduce

$$\begin{aligned}&\mathbb{E}\big[\|X_{\epsilon,\delta,R}(t)\|^q_{\mathcal{W}}\big]\\ &\quad\le C(q,V_{\mathrm{per}})\Bigg\{\|S(t)X_0\|^q_{\mathcal{W}}+\mathbb{E}\Bigg[\bigg(\int_0^t\|X_{\epsilon,\delta,R}(s)\|_{\mathcal{W}}\mathrm{d}s\bigg)^q\Bigg]\\ &\qquad+\mathbb{E}\Bigg[\bigg\|\int_0^t\mathbf{1}_{[0,\tau_R]}(s)S(t-s)B_{N,\delta}(X_{\epsilon,\delta,R}(s))\mathrm{d}W_{\mathrm{per},\epsilon}\bigg\|^q_{\mathcal{W}}\Bigg]\Bigg\}\\ &\quad\le C(q,V_{\mathrm{per}})\Bigg\{\|X_0\|^q_{\mathcal{W}}+\mathbb{E}\Bigg[\bigg(\int_0^t\|X_{\epsilon,\delta,R}(s)\|_{\mathcal{W}}\mathrm{d}s\bigg)^q\Bigg]\\ &\qquad+\mathbb{E}\Bigg[\int_0^t\|B_{N,\delta}(X_{\epsilon,\delta,R}(s))\|^2_{L_2^0(\mathcal{W})}\,\mathrm{d}s\Bigg]^{q/2}\Bigg\}\end{aligned} \tag{72}$$

$$\leq C(q,T,V_{\text{per}})\left\{\|X_0\|^q_{\mathcal{W}} + \int_0^t \mathbb{E}\big[\|X_{\epsilon,\delta,R}(s)\|^q_{\mathcal{W}}\big]\mathrm{d}s + C(\sigma,\delta)M^q(\epsilon,N)\mathbb{E}\left[\int_0^t (1+\|X_{\epsilon,\delta,R}(s)\|^q_{\mathcal{W}})\,\mathrm{d}s\right]\right\}$$

$$\leq g_1 + \int_0^t g_2\mathbb{E}\big[\|X_{\epsilon,\delta,R}(s)\|^q_{\mathcal{W}}\big]\mathrm{d}s, \tag{73}$$

where $g_1 := C(q,T,V_{\text{per}})\|X_0\|^q_{\mathcal{W}} + TC(\sigma,\delta)M^q(\epsilon,N)$ and $g_2 := C(T,q) + C(\sigma,\delta)$ $M^q(\epsilon,N)$. The definition of $X_{\epsilon,\delta,R}$ implies that (72) is finite, hence so is $\mathbb{E}[\|X_{\epsilon,\delta,R}(t)\|^q_{\mathcal{W}}]$. We use Gronwall's lemma in (73) to conclude

$$\mathbb{E}\big[\|X_{\epsilon,\delta,R}(t)\|^q_{\mathcal{W}}\big] \leq \{C(q,T)\|X_0\|^q_{\mathcal{W}} + TC(\sigma,\delta)M^q(\epsilon,N)\}\, e^{C(T,q)+C(T,\sigma,\delta)M^q(\epsilon,N)} \quad \text{for all } t\in[0,T]. \tag{74}$$

The integrability property (71) implies that $\tau_R(\omega) = T$ for $R \geq R(\omega)$, $\mathbb{P}$-a.s. As a result, we deduce

$$\lim_{R\to+\infty} X_{\epsilon,\delta,R}(t) = X_{\epsilon,\delta}(t) \text{ in } \mathcal{W}, \quad t\in[0,T], \quad \mathbb{P}\text{-a.s.}$$

We use Fatou's lemma and we obtain

$$\begin{aligned}\mathbb{E}\big[\|X_{\epsilon,\delta}(t)\|^q_{\mathcal{W}}\big] &\leq \liminf_{R\to+\infty}\mathbb{E}\big[\|X_{\epsilon,\delta,R}(t)\|^q_{\mathcal{W}}\big]\\ &\leq \{C(q,T)\|X_0\|^q_{\mathcal{W}} + TC(\sigma,\delta)M^q(\epsilon,N)\}\, e^{C(T,q)+C(T,\sigma,\delta)M^q(\epsilon,N)}\\ &\qquad \text{for all } t\in[0,T].\end{aligned}$$

Taking the supremum in time finally yields the result. ∎

We obtained (70) by using the càdlàg property of the solution $X_{\epsilon,\delta}$. This allows us to consider an arbitrary $q>2$. If we only relied on the definition of mild solution (see in particular (67)), the exponent $q=2$ would be the maximum exponent we could take. This is exactly the case for the proof of uniqueness in [7, Theorem 7.2], from which we adapted the proof of Proposition 4.9. The proof of [7, Theorem 7.2, (7.6)], which is exactly our (70), relies on a fixed point argument instead. We cannot use this argument, since we lack the global Lipschitz property for the stochastic integrand $B_{N,\delta}$. The need for $q>2$, and not simply $q=2$, is motivated by [7, Proposition 7.3], which we will use in the next section.

4.6. Small-noise regime analysis. In this subsection, we investigate the small-noise regime analysis for solutions $X_{\epsilon,\delta}$ to (66).

PROPOSITION 4.10. *Let the hypotheses of Proposition* 4.7 *be satisfied. In addition, assume the following scaling for* ϵ, N*:*

$$N\epsilon^{\theta} \geq 1 \qquad \textit{for some } \theta > 7. \tag{75}$$

For fixed $\delta>0$, $T>0$, $r>0$, $q>2$, *we have*

$$\lim_{\epsilon\downarrow 0}\mathbb{P}\left(\sup_{t\in[0,T]}\|X_{\epsilon,\delta}(t)-Z(t)\|^q_{\mathcal{W}} \geq r\right) = 0,$$

where Z *is the unique (deterministic) solution of*

$$\begin{cases} dZ(t) = (AZ(t) + \alpha_{\text{per}}Z(t))\,dt,\\ Z(0) = X_0.\end{cases} \tag{76}$$

Proof. We adapt the proof of [7, Proposition 12.1]. The scaling (75) implies that $M(\epsilon, N) \to 0$ in the simultaneous limit of ϵ and N. We write

$$X_{\epsilon,\delta}(t) - Z(t) = \int_0^t S(t-s)\alpha_{\mathrm{per}}(X_{\epsilon,\delta}(s) - Z(s))\mathrm{d}s + \int_0^t S(t-s)B_{N,\delta}(X_{\epsilon,\delta}(s))\mathrm{d}W_{\mathrm{per},\epsilon}.$$

We use [7, Proposition 7.3] and Proposition 4.9 to deduce

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]} \|X_{\epsilon,\delta}(s) - Z(s)\|^q_{\mathcal{W}}\right] \\
&\quad\le C(T,q,V_{\mathrm{per}})\mathbb{E}\left[\int_0^t \|X_{\epsilon,\delta}(u) - Z(u)\|^q_{\mathcal{W}}\,\mathrm{d}u\right] \\
&\qquad + \mathbb{E}\left[\sup_{s\in[0,T]} \left\|\int_0^s S(t-s)B_{N,\delta}(X_{\epsilon,\delta})\mathrm{d}W_{\mathrm{per},\epsilon}\right\|^q\right] \\
&\quad\le C(T,q,V_{\mathrm{per}})\mathbb{E}\left[\int_0^t \|X_{\epsilon,\delta}(u) - Z(u)\|^q_{\mathcal{W}}\,\mathrm{d}u\right] \\
&\qquad + C(\sigma,\delta,T,q)M^q(\epsilon,N)\mathbb{E}\left[\int_0^T (1+\|X_{\theta,\delta}\|^q_{\mathcal{W}})\mathrm{d}s\right] \qquad (77)\\
&\quad\le C(T,q,V_{\mathrm{per}})\int_0^t \mathbb{E}\left[\sup_{s\in[0,u]} \|X_{\epsilon,\delta}(u) - Z(u)\|^q_{\mathcal{W}}\right]\mathrm{d}u \\
&\qquad + C(\sigma,\delta,T,q)M^q(\epsilon,N)T(1+\Theta), \qquad (78)
\end{aligned}$$

where Θ is defined in Proposition 4.9. Thanks to the same proposition, (77) is finite. The scaling (75) also implies that Θ is bounded in ϵ, N. We can apply the Gronwall inequality to (78) to deduce that

$$\mathbb{E}\left[\sup_{s\in[0,T]} \|X_{\epsilon,\delta}(s) - Z(s)\|^q_{\mathcal{W}}\right] \le C(\sigma,\delta,T,q)M^q(\epsilon,N)T(1+\theta)e^{C(T,q,V_{\mathrm{per}})} \to 0$$
$$\text{as } \epsilon \to 0,\ N \to \infty.$$

Chebyshev's inequality gives the result. ∎

The prescribed scaling in N, ϵ stated in Proposition 4.10 is compatible with the scalings of Propositions 1.1 and 1.2 and Theorem 1.3. See also Remark 3.2.

4.7. Main existence and uniqueness result. We now turn to the key existence and uniqueness result for the regularized Dean–Kawasaki model (63) or equivalently (14).

Remark 4.11. Let us fix $\eta > \delta > 0$. We first notice that, for a deterministic initial condition $X_0 = (\rho_0, j_0) \in \mathcal{W}$ such that (68) is satisfied, there exists $T = T(X_0) \in (0, \infty]$ such that the solution Z to (76) satisfies

$$Z(t,x) \ge \delta + (\eta - \delta)/2 \qquad \text{for all } x \in D, \ \text{ for all } t \in [0,T).$$

This is implied by the time-continuity of Z with respect to the $\mathcal{W}$-norm and by Proposition 4.1.

Proof of Theorem 1.6. Fix δ so that $0 < \delta < \eta$ and consider $T(X_0)$ as indicated in Remark 4.11. Proposition 4.7 provides existence of a solution $X_{\epsilon,\delta}$ to (66). For some $q > 2$, we rely on Proposition 4.1 and write

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T(X_0)]}\|X_{\epsilon,\delta}(t)-Z(t)\|_{C^0_{\mathrm{per}}(D)\times C^0_{\mathrm{per}}(D)}\geq\frac{\eta-\delta}{2}\right)\\
&\quad=\mathbb{P}\left(\sup_{t\in[0,T(X_0)]}\|X_{\epsilon,\delta}(t)-Z(t)\|^q_{C^0_{\mathrm{per}}(D)\times C^0_{\mathrm{per}}(D)}\geq\frac{(\eta-\delta)^q}{2^q}\right)\\
&\quad\leq\mathbb{P}\left(\sup_{t\in[0,T(X_0)]}\|X_{\epsilon,\delta}(t)-Z(t)\|^q_{\mathcal{W}}\geq\frac{C^{-q}(\eta-\delta)^q}{2^q}\right)\leq\nu,
\end{aligned}$$

where the last inequality holds for ϵ small enough (or equivalently N big enough), thanks to Proposition 4.10. It follows that

$$\mathbb{P}\left(X_{\epsilon,\delta}(x,t)\geq\delta \text{ for all } t\in[0,T(X_0)),\text{ for all } x\in D\right)\geq 1-\nu.$$

This implies that $\mathbb{P}\left(B_{N,\delta}(X_{\epsilon,\delta})=B_N(X_{\epsilon,\delta}) \text{ for all } t\in[0,T(X_0))\right)\geq 1-\nu$. We take $X_\epsilon:=X_{\epsilon,\delta}$ and employ the existence and uniqueness results from Proposition 4.7 to conclude the proof. ∎

The dependence of T on X_0 is yet to be properly investigated. In the special case of constant initial data $X_0=(\rho_0,j_0)=(C,0)$, for some $C>\delta>0$, the solution is stationary, hence we can pick any finite $T(X_0)$.

Remark 4.12. We have relied on scalings of type $N\epsilon^\theta=1$ (or $N\epsilon^\theta\geq 1$), for some $\theta>0$, to prove several results throughout the paper. Some of these scalings could be improved (i.e., θ could be lowered) in at least two points, specifically:

(a) *Tightness of* $\{\rho_\epsilon\}_\epsilon$, *Proposition* 1.1. We relied on the compact embedding $H^1(D)\subset L^2(D)$ to show that the initial conditions $\{\rho_\epsilon(\cdot,0)\}_\epsilon$ are tight in L^2. If one uses the compact embedding $H^{1/2+\delta/2}(D)\subset L^2(D)$ instead, for some $\delta\in(0,1)$, the scaling is less demanding, as $\|w_\epsilon(\cdot)\|_{H^{1/2+\delta/2}}\propto\epsilon^{-2-\delta}$.

In addition, the time-regularity estimate can be improved by computing the expectation first in the second-to-last inequality of (18). In this case, the estimate proceeds with the bound

$$\underbrace{1-\frac{\sqrt{4\pi\epsilon^2}}{\sqrt{2\pi(2\epsilon^2+V_{s,t})}}}_{=:T_1}+\underbrace{\frac{\sqrt{4\pi\epsilon^2}}{\sqrt{2\pi(2\epsilon^2+V_{s,t})}}\left(1-\exp\left\{-\frac{\mu^2_{s,t}}{2(2\epsilon^2+V_{s,t})}\right\}\right)}_{=:T_2},$$

where

$$\mu_{s,t}:=\mathbb{E}\left[q(t)-q(s)\right]\leq C|t-s|,\qquad V_{s,t}:=\mathrm{Var}\left[q(t)-q(s)\right]\leq C|t-s|^2.$$

It is not difficult to bound T_1 and T_2 by $C\epsilon^{-1-\delta}|t-s|^{1+\delta}$, where δ can be chosen in $(0,1]$. Overall, the scaling $N\epsilon^{2+\delta}$, for some $\delta\in(0,1]$, is sufficient to provide tightness of $\{\rho_\epsilon\}_\epsilon$. We believe that similar arguments could be applied to $\{j_\epsilon\}_\epsilon$ and $\{j_{2,\epsilon}\}_\epsilon$ as well.

(b) *Functional setting of section 4*. If we redefine $\mathcal{W}$ as $\mathcal{W}:=H^{1/2+\delta/2}_{\mathrm{per}}(D)\times H^{1/2+\delta/2}_{\mathrm{per}}(D)$, this could lead to a better scaling in Lemma 4.3, for reasons analogous to point (a). This would then lead to a better scaling in Theorem 1.6.

Appendix A. Gaussian tools. This appendix is devoted to a concise exposition of a few useful facts concerning Gaussian random variables.

Definition A.1. *A Gaussian random vector X with mean $\mu\in\mathbb{R}^d$ and covariance matrix Σ, denoted as $X\sim\mathcal{N}(\mu,\Sigma)$, has the probability density function given by*

$\mathcal{G}(x,\mu,\Sigma) = \det(2\pi\Sigma)^{-1/2}\exp\left\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right\}$. *In the real-valued case, i.e., for X of mean μ and variance σ^2, the above is simply*

$$\mathcal{G}(x,\mu,\sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$

Lemma A.2 (Fourier transform for Gaussians). *The Fourier transform of an $\mathbb{R}^d$-valued Gaussian random vector $Y \sim \mathcal{N}(\mu,\Sigma)$ is given by*

$$\mathbb{R}^d \ni \xi \mapsto \mathbb{E}\left[e^{-i\langle \xi, Y\rangle}\right] = \exp\left\{-i\langle\mu,\xi\rangle - \frac{1}{2}\langle\xi,\Sigma\xi\rangle\right\}.$$

Lemma A.3 (conditional law for Gaussian vectors). *Let $b\in\mathbb{R}$. For a bivariate Gaussian random vector $Y=(Y_1,Y_2)$, the conditional density of Y_1 given $Y_2=b$ is*

$$f_{Y_1|Y_2}(y_1|Y_2=b) = \mathcal{G}\left(y_1, \mu_{Y_1} + \frac{\sigma_{Y_1}}{\sigma_{Y_2}}\chi(b-\mu_{Y_2}), (1-\chi^2)\sigma_{Y_1}^2\right),$$

where $\chi = Corr(Y_1,Y_2)$.

Lemma A.2 can be found in [17, Chapter 16], and Lemma A.3 can be found in [4, section 4.7].

Lemma A.4 (multiplication of Gaussian kernels). *Given $f(x) := \mathcal{G}(x,\mu_f,\sigma_f^2)$ and $g(x) := \mathcal{G}(x,\mu_g,\sigma_g^2)$, we have the multiplication rule*

$$f(x)g(x) = \mathcal{G}(x,\mu_{fg},\sigma_{fg}^2)\frac{1}{\sqrt{2\pi(\sigma_f^2+\sigma_g^2)}}\exp\left\{-\frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)}\right\},$$

where we have set

$$\mu_{fg} := \frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}, \qquad \sigma_{fg}^2 := \frac{\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2}.$$

Lemma A.5 (moments of Gaussian random variables). *Let $X\sim\mathcal{N}(\mu,\sigma^2)$. For $n\in\mathbb{N}$, we have*

$$\begin{aligned} M(n,\mu,\sigma^2) &:= \mathbb{E}\left[|X|^n\right] \le C(n)\left\{\mu^n + \sigma^n(n-1)!!\right\},\\ m(n,\mu,\sigma^2) &:= \mathbb{E}\left[X^n\right] = \sum_{j\in\mathbb{N},\, 2j\le n}(2j-1)!!\binom{n}{2j}\sigma^{2j}\mu^{n-2j}, \end{aligned}$$

where $n!! := \sum_{k=0}^{\lceil n/2\rceil-1}(n-2k)$, for $n\in\mathbb{N}$.

Lemma A.5 can be proved by induction on n, by splitting X as $(X-\mu)+\mu$ and using the results for moments of zero-mean Gaussian random variables. Lemma A.4 follows from simple algebraic computations.

Lemma A.6 (Ornstein–Uhlenbeck process). *Let $A,\Sigma\in\mathbb{R}^{2\times 2}$, and let W be a bivariate Brownian motion. For any $t\in[0,T]$, set $\Phi(t) := e^{At}$.*

(i) *The stochastic equation*

$$dX(t) = AX(t)dt + \Sigma dW(t), \qquad X(0) = X_0, \tag{79}$$

has a unique solution $X(t) = (X_1(t),X_2(t))$ explicitly given by

$$X(t) = \Phi(t)X_0 + \Phi(t)\int_0^t \Phi^{-1}(s)\Sigma dW(s). \tag{80}$$

(ii) *If X_0 is a Gaussian random vector independent of W, then $X(t)$ is a Gaussian random vector for any $t \in [0,T]$.*

(iii) *With the same assumption as in* (ii)*, if in addition $Cov(X_0, X_0)$ is positive definite, then there exists $\nu > 0$ such that $Var(X^1(t)) \geq \nu$ and $Var(X^2(t)) \geq \nu$ for any $t \in [0,T]$.*

(iv) *With the same assumption as in* (iii)*, the following quantities are Lipschitz on $[0,T]$: the mean of $X_1(t)$ and $X_2(t)$, the variance of $X_1(t)$ and $X_2(t)$, the correlation between $X_1(t)$ and $X_2(t)$.*

Proof. Part (i). Existence and uniqueness of a solution is granted by [27, Theorem 5.2.1]. It is straightforward to see that (80) is indeed the solution by computing the Itô-differential of $X(t)$.

Part (ii). The integrand $\Phi^{-1}(s)\Sigma$ being deterministic, $\Phi(t)\int_0^t \Phi^{-1}(s)\Sigma \mathrm{d}W(s)$ is a Gaussian process. In addition, $\Phi(t)X_0$ is a Gaussian vector by linearity. Stochastic independence of X_0 and W grants that the sum of the aforementioned two vectors is a Gaussian vector.

Part (iii). Thanks to the independence of W and X_0, we can limit ourselves to studying $\mathrm{Cov}(\Phi(t)X_0, \Phi(t)X_0)$. We observe that

$$\mathrm{Cov}(\Phi(t)X_0, \Phi(t)X_0) = \Phi(t)\mathrm{Cov}(X_0, X_0)\Phi^T(t) =: B(t).$$

Since $\mathrm{Cov}(X_0, X_0)$ is definite positive, this entails that the continuous function $t \mapsto y^T B(t) y$ is strictly positive on $[0,T]$ for any given $y \in \mathbb{R}^2 \setminus \{(0,0)\}$. The claim then follows by taking $y = (1,0)$ and $y = (0,1)$.

Part (iv). We notice that

$$\left\|\mathbb{E}\big[X(t) - X(s)\big]\right\| = \left\|\mathbb{E}\big[(\Phi(t) - \Phi(s))\, X_0\big]\right\| \leq C(A)\mathbb{E}\big[\|X_0\|\big]|t-s|,$$

and the Lipschitz property for the mean of $X_1(t)$ and $X_2(t)$ is settled. As for the variances, we compute

$$\begin{aligned}\mathrm{Cov}(X(t), X(t)) - \mathrm{Cov}(X(s), X(s)) = {} & \Phi(t)\left[\int_0^t \Phi^{-1}(u)\Sigma\Sigma^T\Phi^{-T}(u)\mathrm{d}u\right]\Phi^T(t) \\ & - \Phi(s)\left[\int_0^s \Phi^{-1}(u)\Sigma\Sigma^T\Phi^{-T}(u)\mathrm{d}u\right]\Phi^T(s) \\ & + \Phi(t)\mathrm{Cov}(X_0, X_0)\Phi^T(t) \\ & - \Phi(s)\mathrm{Cov}(X_0, X_0)\Phi^T(s),\end{aligned} \tag{81}$$

and the Lipschitz property for the variance of $X_1(t)$ and $X_2(t)$ follows from the Lipschitz property for $\Phi(t)$ and $\int_0^t \Phi^{-1}(u)\Sigma\Sigma^T\Phi^{-T}(u)\mathrm{d}u$. As for the correlation between $X_1(t)$ and $X_2(t)$, the Lipschitz property can be derived by using the definition

$$\mathrm{Corr}(X_1(t), X_2(t)) := \frac{\mathrm{Cov}(X_1(t), X_2(t))}{\sqrt{\mathrm{Var}(X_1(t))\mathrm{Var}(X_2(t))}}$$

and observing that $\mathrm{Var}(X_1(t))$, $\mathrm{Var}(X_2(t))$ are bounded away from 0 (by A.6) and that $\mathrm{Var}(X_1(t))$, $\mathrm{Var}(X_2(t))$, $\mathrm{Cov}(X_1(t), X_2(t))$ are Lipschitz by (81). ∎

Appendix B. Auxiliary tools. We list and prove some auxiliary tools used repeatedly in the proofs of the main results of section 3. We start with time regularity of Gaussian moments, under Assumption (G), in subsection B.1. We deal with time regularity for the Fokker–Planck equation (15) under Assumption (NG) in subsection B.2. We estimate the second moment of $\rho_\epsilon^{-1}(x,t)$, where $\rho_\epsilon(x,t)$ is defined in (7), giving a proof for both Assumption (G) and Assumption (NG), in subsection B.3.

B.1. Time regularity of specific Gaussian moments.

LEMMA B.1. *Let $T > 0$, $n \in \mathbb{N}$, $c \geq 2$, $\nu > 0$ be real numbers. Let $\mu, \sigma^2 \colon [0,T] \to \mathbb{R}$ be Lipschitz functions, with Lipschitz constant L. Let $\mathcal{Q}_{n,t}(x)$ be a polynomial of degree n in x, and Lipschitz coefficients in t, again with Lipschitz constant L. Assume that $\sigma^2(t) \geq \nu$ for all $t \in [0,T]$. Then there exists $\beta > 0$ such that*

$$\int_{\mathbb{R}} \left|\mathcal{Q}_{n,t}(x)\mathcal{G}(x,\mu(t),\sigma^2(t)) - \mathcal{Q}_{n,s}(x)\mathcal{G}(x,\mu(s),\sigma^2(s))\right|^c dx \leq C|t-s|^{1+\beta} \qquad \text{for all } s,t \in [0,T]$$

for a constant $C = C(T,\nu,L,c)$. In addition, if $p = 0$, $c = 2$, and $\mathcal{Q}_{n,t}$ is a constant, then $\beta = 1$.

Proof. Because of the general inequality $\left|\sum_{i=0}^n a_i\right|^c \leq (n+1)^c \sum_{i=0}^n |a_i|^c$, it is sufficient to prove the statement for each monomial composing $\mathcal{Q}_{n,t}(x)$. We can thus restrict ourselves to proving the statement with the choice $\mathcal{Q}_{p,t}(x) := A(t)x^p$, for any $p \in \mathbb{N}$, and where A is Lipschitz with constant L.

We add and subtract relevant quantities in the integral we have to compute. As a result we get

$$\begin{aligned}
&\int_{\mathbb{R}} \left|A(t)x^p\mathcal{G}(x,\mu(t),\sigma^2(t)) - A(s)x^p\mathcal{G}(x,\mu(s),\sigma^2(s))\right|^c \mathrm{d}x \\
&\quad \leq 2^c \underbrace{\int_{\mathbb{R}} \left|(A(t)-A(s))x^p\mathcal{G}(x,\mu(t),\sigma^2(t))\right|^c \mathrm{d}x}_{=:T_1} \\
&\qquad + 2^c \underbrace{\int_{\mathbb{R}} \left|A(s)x^p\left(\mathcal{G}(x,\mu(t),\sigma^2(t)) - \mathcal{G}(x,\mu(s),\sigma^2(s))\right)\right|^c \mathrm{d}x}_{=:T_2}.
\end{aligned}$$

We estimate T_1, T_2 separately. Since A is Lipschitz and σ^2 is bounded from below, we obtain

$$\begin{aligned}
T_1 &\leq L^c|t-s|^c \int_{\mathbb{R}} |x|^{cp}\mathcal{G}(x,\mu(t),\sigma^2(t))^c \mathrm{d}x \\
&= \frac{L^c}{c^{1/2}(2\pi\sigma^2(t))^{(c-1)/2}} M\left(cp, \mu(t), \frac{\sigma^2(t)}{c}\right)|t-s|^c \\
&\leq \frac{L^c}{c^{1/2}(2\pi\nu)^{(c-1)/2}} C(T,p,c)|t-s|^c \leq C|t-s|^c,
\end{aligned}$$

where we have also relied on Lemmas A.4 and A.5. In order to estimate T_2, we rewrite the integral as

$$\begin{aligned}
&\int_{\mathbb{R}} |A|^c(s)|x|^{cp}\left|\mathcal{G}(x,\mu(t),\sigma^2(t)) - \mathcal{G}(x,\mu(s),\sigma^2(s))\right|^\alpha \\
&\qquad \cdot \left|\mathcal{G}(x,\mu(t),\sigma^2(t)) - \mathcal{G}(x,\mu(s),\sigma^2(s))\right|^{c-\alpha} \mathrm{d}x
\end{aligned} \tag{82}$$

for some $\alpha \in (c-2, c-1)$. We apply the Hölder inequality with conjugate exponents $\frac{2}{c-\alpha}$ and $\frac{2}{2-c+\alpha}$ and obtain

$$T_2 \leq \Big(\int_{\mathbb{R}} |A|^{2c/(2-c+\alpha)}(s)|x|^{2pc/(2-c+\alpha)}\big|\mathcal{G}(x,\mu(t),\sigma^2(t)) - \mathcal{G}(x,\mu(s),\sigma^2(s))\big|^{2\alpha/(2-c+\alpha)}\,\mathrm{d}x\Big)^{(\alpha+2-c)/2} \times \left(\int_{\mathbb{R}} \left|\mathcal{G}(x,\mu(t),\sigma^2(t)) - \mathcal{G}(x,\mu(s),\sigma^2(s))\right|^2 \mathrm{d}x\right)^{\frac{c-\alpha}{2}}.$$

The first term can be controlled using the boundedness of A and Lemmas A.4 and A.5, similarly to the argument for T_1. We get

$$\begin{aligned}
&\Big(\int_{\mathbb{R}} |A|^{2c/(2-c+\alpha)}(s)|x|^{2pc/(2-c+\alpha)}\big|\mathcal{G}(x,\mu(t),\sigma^2(t)) - \mathcal{G}(x,\mu(s),\sigma^2(s))\big|^{2\alpha/(2-c+\alpha)}\,\mathrm{d}x\Big)^{(\alpha+2-c)/2}\\
&\leq C(A,c,p,\nu)\left\{M\left(\frac{2pc}{2-c+\alpha},\mu(t),\frac{\sigma^2(t)(2-c+\alpha)}{2\alpha}\right) + M\left(\frac{2pc}{2-c+\alpha},\mu(s),\frac{\sigma^2(s)(2-c+\alpha)}{2\alpha}\right)\right\}\\
&\leq C(A,c,p,\nu,\alpha).
\end{aligned}$$

As for the second term of the product bounding T_2, we rely on Fourier analysis and Taylor expansions. More precisely, we rely on Parseval's equality, Lemma A.2, and some simple rearrangement to write

$$\begin{aligned}
&\int_{\mathbb{R}} \left|\mathcal{G}(x,\mu(t),\sigma^2(t)) - \mathcal{G}(x,\mu(s),\sigma^2(s))\right|^2 \mathrm{d}x\\
&= C\int_{\mathbb{R}} \left|e^{-i\mu(t)\xi-\frac{1}{2}\sigma^2(t)\xi^2} - e^{-i\mu(s)\xi-\frac{1}{2}\sigma^2(s)\xi^2}\right|^2 \mathrm{d}\xi\\
&\leq C\int_{\mathbb{R}} \left|\left\{e^{-i\mu(t)\xi} - e^{-i\mu(s)\xi}\right\}e^{-\frac{1}{2}\sigma^2(t)\xi^2}\right|^2 \mathrm{d}\xi + C\int_{\mathbb{R}} \left|e^{-i\mu(s)\xi}\left\{e^{-\frac{1}{2}\sigma^2(t)\xi^2} - e^{-\frac{1}{2}\sigma^2(s)\xi^2}\right\}\right|^2 \mathrm{d}\xi\\
&\leq C\underbrace{\int_{\mathbb{R}} \left|\left\{e^{-i\mu(t)\xi} - e^{-i\mu(s)\xi}\right\}e^{-\frac{1}{2}\sigma^2(t)\xi^2}\right|^2 \mathrm{d}\xi}_{=:T_3} + C\underbrace{\int_{\mathbb{R}} \left|e^{-\frac{1}{2}\sigma^2(t)\xi^2} - e^{-\frac{1}{2}\sigma^2(s)\xi^2}\right|^2 \mathrm{d}\xi}_{=:T_4}.
\end{aligned}$$

For T_3, we use the mean-value theorem applied to the map $y \mapsto e^{iy}$ and the Lipschitz properties of μ to deduce

$$T_3 \leq L^2|t-s|^2\int_{\mathbb{R}} \xi^2 e^{-\sigma^2(t)\xi^2}\mathrm{d}\xi = L^2|t-s|^2\sqrt{2\pi}\left[\frac{1}{\sigma(t)^2}\right]^{3/2} \leq C(L,\nu)|t-s|^2,$$

where we have used the definition of the Gaussian kernel and the bound $\sigma^2(t) \geq \nu$. We move on to T_4. We rely on Lemma A.4 and we expand the square in the integrand to deduce

$$T_4 = \sqrt{\frac{\pi}{\sigma^2(t)}} + \sqrt{\frac{\pi}{\sigma^2(s)}} - 2\sqrt{\frac{2\pi}{\sigma^2(t)+\sigma^2(s)}} \leq C(\nu)\left|\sigma^2(t)-\sigma^2(s)\right|^2 \leq C(\nu)|t-s|^2.$$

The second inequality above is the Lipschitz property of σ^2, while the first inequality is justified by the midpoint estimate $f(\sigma^2(t)) + f(\sigma^2(s)) - 2f([\sigma^2(t)+\sigma^2(s)]/2) \leq C(\nu)\left|\sigma^2(t)-\sigma^2(s)\right|^2$ for the function $f\colon [\nu,\infty) \to \mathbb{R}\colon y \mapsto \sqrt{\pi/y}$. Such expansion is a consequence of the superposition of the second-order Taylor expansions (with Lagrange remainder) of $f(\sigma^2(t))$ and $f(\sigma^2(s))$ centered around $[\sigma^2(t)+\sigma^2(s)]/2$. Putting T_3 and T_4 together, we deduce

$$\left(\int_{\mathbb{R}} \left|\mathcal{G}(x,\mu(t),\sigma^2(t)) - \mathcal{G}(x,\mu(s),\sigma^2(s))\right|^2 \mathrm{d}x\right)^{\frac{c-\alpha}{2}} \leq C|t-s|^{2\cdot\frac{c-\alpha}{2}} = C|t-s|^{c-\alpha}.$$

We rename $\beta := c-\alpha-1 \in (0,1)$. We combine the above estimates and we obtain

$$\int_{\mathbb{R}} \left|A(t)x^p\mathcal{G}(x,\mu(t),\sigma^2(t)) - A(s)x^p\mathcal{G}(x,\mu(s),\sigma^2(s))\right|^c \mathrm{d}x \leq C|t-s|^{1+\beta},$$

as desired. If $p=0$, $c=2$, and $\mathcal{Q}_{n,t}$ is a constant, then $\beta=1$. This is because $T_1=0$, and one may simply take $\alpha=0$ in (82). ∎

LEMMA B.2. *Let $X \sim \mathcal{N}(\mu,\sigma^2)$ and let $x \in \mathbb{R}$. Then*

(83)
$$\mathbb{E}\left[w_\epsilon(x-X)X^n\right] = \mathcal{G}(x,\mu,\epsilon^2+\sigma^2)\cdot m\left(n, \frac{x\sigma^2+\mu\epsilon^2}{\epsilon^2+\sigma^2}, \frac{\epsilon^2\sigma^2}{\epsilon^2+\sigma^2}\right), \quad n \in \mathbb{N}\cup\{0\}.$$

(84)
$$\mathbb{E}\left[w'_\epsilon(x-X)X^n\right] = \frac{\mathcal{G}(x,\mu,\epsilon^2+\sigma^2)}{\epsilon^2}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}m\left(k+1, \frac{(\mu-x)\epsilon^2}{\epsilon^2+\sigma^2}, \frac{\epsilon^2\sigma^2}{\epsilon^2+\sigma^2}\right),$$
$$n \in \mathbb{N}\cup\{0\}.$$

$$\mathbb{E}\left[w''_\epsilon(x-X)X^n\right] = \mathcal{G}(x,\mu,\epsilon^2+\sigma^2)\sum_{k=0}^{n}\binom{n}{k}x^{n-k}\left\{-\frac{1}{\epsilon^2}m\left(k, \frac{(\mu-x)\epsilon^2}{\epsilon^2+\sigma^2}, \frac{\epsilon^2\sigma^2}{\epsilon^2+\sigma^2}\right)\right.$$
(85)
$$\left. + \frac{1}{\epsilon^4}m\left(k+2, \frac{(\mu-x)\epsilon^2}{\epsilon^2+\sigma^2}, \frac{\epsilon^2\sigma^2}{\epsilon^2+\sigma^2}\right)\right\}.$$

The proof of Lemma B.2 is a straightforward application of multiplication properties for Gaussian kernels and Gaussian moments, as stated in Lemmas A.4 and A.5.

Remark B.3. It is worth noticing that the right-hand sides of (83), (84), and (85) satisfy the requirements of Lemma B.1. To see this, we notice that

$$m\left(n, \frac{x\sigma^2+\mu\epsilon^2}{\epsilon^2+\sigma^2}, \frac{\epsilon^2\sigma^2}{\epsilon^2+\sigma^2}\right)$$

is a polynomial of degree n (with ϵ-dependent coefficients) in the variable x. For time-dependent $\mu(t)$, $\sigma^2(t)$ satisfying the hypotheses of Lemma B.1, it follows that $\epsilon^2+\sigma^2 \geq \nu > 0$ for any $\epsilon > 0$. These facts imply that the right-hand side of (83) can be written in the form $\mathcal{Q}_{\epsilon,n,t}(x)\mathcal{G}(x,\mu(t),\sigma^2(t)+\epsilon^2)$, where the polynomial $\mathcal{Q}_{\epsilon,n,t}(x)$ has time-Lipschitz coefficients whose Lipschitz constants are uniformly bounded as $\epsilon \to 0$. For these reasons, (83) satisfies the statement of Lemma B.1, and the result of the application of Lemma B.1 on (83) is independent of ϵ as $\epsilon \to 0$. On a similar note, we notice that

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} m\left(k+1; \frac{(\mu - x)\epsilon^2}{\epsilon^2+\sigma^2}, \frac{\epsilon^2\sigma^2}{\epsilon^2+\sigma^2}\right)$$

can be written as $\mathcal{Q}_{\epsilon,n,t}(x) := \epsilon^2 \mathcal{P}_{\epsilon,n,t}(x)$, where the polynomial $\mathcal{P}_{\epsilon,n,t}(x)$ has time-Lipschitz coefficients whose Lipschitz constants are uniformly bounded as $\epsilon \to 0$. This is a consequence of the Gaussian moments of order at least one, for a Gaussian kernel with both mean $\frac{(\mu-x)\epsilon^2}{\epsilon^2+\sigma^2}$ and variance $\frac{\epsilon^2\sigma^2}{\epsilon^2+\sigma^2}$ featuring a multiplicative factor ϵ^2. This factor can be canceled out with that appearing in the right-hand side of (84), which can hence be written in the form $\mathcal{P}_{\epsilon,n,t}(x)\mathcal{G}(x, \mu(t), \sigma^2(t) + \epsilon^2)$. For these reasons, (84) satisfies the statement of Lemma B.1, and the result of the application of Lemma B.1 on (84) is independent of ϵ as $\epsilon \to 0$. Similar considerations apply for (85). The contents of this remark apply under Assumption (G) for the time-dependent X being precisely the Langevin particle $q_i(t)$ satisfying (2).

In addition, the right-hand sides of (83), (84), and (85) are Lipschitz in time, with Lipschitz constant independent of ϵ (see the discussion above) and x (each one of the right-hand sides being a product of a polynomial with a decaying exponential).

B.2. Fokker–Planck time regularity in the case of nonvanishing potential $\boldsymbol{V}$. The contents of this subsection should be seen as the "replacement" of Lemma B.1, Lemma B.2, and Remark B.3 under Assumption (NG). We consider the *Fokker–Planck* equation associated with (2), namely,

$$\begin{cases} \dfrac{\partial g}{\partial t} = -\nabla \cdot (g\mu) + \dfrac{\sigma^2}{2}\dfrac{\partial^2}{\partial p^2} g, \\ g(0,p,q) = g_0(p,q), \end{cases} \tag{86}$$

where $g_0(p,q)$ is the law of $(q(0), p(0))$.

Remark B.4. We comment on some consequences of [15, Theorem 0.1]. This result, among many things, implies the following bound for the solution to (15):

$$\|\overline{g}(t,\cdot,\cdot)\|_{M^{1/2}H^{s,s}} \le C(1 + Q_s(t))e^{-\tau t}\|\overline{g}_0\|_{M^{1/2}H^{-s,-s}}, \tag{87}$$

where $\tau > 0$, where $C = C(\gamma, \sigma, V, \tau,)$, and $Q_s(t)$ is a continuous positive function such that $\lim_{t\to 0^+} Q_s(t) = +\infty$, $\lim_{t\to+\infty} Q_s(t) < +\infty$, and where $M^{1/2}H^{s,s}$ denotes the weighted isotropic Sobolev space of order s with weight $M^{-1/2}$, as stated in Assumption (NG). In addition, well-posedness of (15) is proved in $M^{1/2}\mathcal{S}'(\mathbb{R}^{2d})$. The auxiliary initial condition $\overline{g}_0$ mentioned in Assumption (NG) may be used in (87) to deduce that

$$\|\overline{g}(s,\cdot,\cdot)\|_{M^{1/2}H^{5,5}} \le C_{\overline{t}} \qquad \text{for all } s \ge \overline{t} > 0. \tag{88}$$

The well-posedness of (15) in $M^{1/2}\mathcal{S}'(\mathbb{R}^{2d})$, the choice of $\overline{g}_0$ made in Assumption (NG), and (88) imply the following bound for the solution to (86):

$$\|g(t,\cdot,\cdot)\|_{M^{1/2}H^{5,5}} = \|\overline{g}(\overline{t} + t\cdot,\cdot)\|_{M^{1/2}H^{5,5}} \le C_{\overline{t}} \qquad \text{for all } t \ge 0. \tag{89}$$

We remind the reader that g is the probability density function of a Langevin particle $(q_i(t), p_i(t))$ satisfying (2).

LEMMA B.5. *Let $g(t,q,p)$ be the solution to* (86), *and let Assumption* (NG) be satisfied. For some $\alpha \in (1/4, 1/2)$ and any $0 \le s < t \le T$, we have

$$\|g(t,\cdot,\cdot) - g(s,\cdot,\cdot)\|_{L^2(\mathbb{R}^2)} \le C|t-s|, \tag{90}$$

$$\|M^{-\alpha}\left(g(t,\cdot,\cdot) - g(s,\cdot,\cdot)\right)\|_{L^\infty(\mathbb{R}^2)} \le C|t-s|, \tag{91}$$

$$\left\|M^{-\alpha}(\partial/\partial q)\left(g(t,\cdot,\cdot) - g(s,\cdot,\cdot)\right)\right\|_{L^\infty(\mathbb{R}^2)} \le C|t-s|. \tag{92}$$

Proof. We write

$$\begin{aligned}
\|g(t,q,p) - g(s,q,p)\|^2_{L^2(\mathbb{R}^2)} &\le 2\left\|\int_s^t -\nabla\cdot(\mu g)\mathrm{d}z\right\|^2_{L^2(\mathbb{R}^2)} + 2\left\|\int_s^t \frac{\sigma^2}{2}\frac{\partial^2}{\partial p^2} g\,\mathrm{d}z\right\|^2_{L^2(\mathbb{R}^2)}\\
&\le 2|t-s|\int_s^t \|\nabla\cdot(\mu)g + \mu\cdot\nabla g\|^2_{L^2(\mathbb{R}^2)}\mathrm{d}z\\
&\quad + 2|t-s|\int_s^t \left\|\frac{\sigma^2}{2}\frac{\partial^2}{\partial p^2}g\right\|^2_{L^2(\mathbb{R}^2)}\mathrm{d}z\\
&\le 2|t-s|\int_s^t \big\|M^{1/2-\alpha}M^{-1/2+\alpha}\left(\nabla\cdot(\mu)g\right.\\
&\quad \left.+ \mu\cdot\nabla g\right)\big\|^2_{L^2(\mathbb{R}^2)}\mathrm{d}z\\
&\quad + 2|t-s|\int_s^t \left\|M^{1/2-\alpha}M^{-1/2+\alpha}\frac{\sigma^2}{2}\frac{\partial^2}{\partial p^2}g\right\|^2_{L^2(\mathbb{R}^2)}\mathrm{d}z.
\end{aligned}\tag{93}$$

Assumption (NG) implies that V has at most polynomial growth, while M decays exponentially in p, q. This immediately implies that $\|\nabla\cdot(\mu)M^{1/2-\alpha}\|_{L^\infty(\mathbb{R}^2)} < \infty$ and $\||\mu|M^{1/2-\alpha}\|_{L^\infty(\mathbb{R}^2)} < \infty$. In addition, $M^{-1/2+\alpha}g$ is uniformly bounded in time in $H^{2,2}(\mathbb{R}^2)$ thanks to (89). This is enough to control the $L^2(\mathbb{R}^2)$-norm of the remaining terms $M^{-1/2+\alpha}g$, $M^{-1/2+\alpha}\nabla g$, $M^{-1/2+\alpha}(\partial^2/\partial p^2)g$, and proceed in (93) to deduce (90). As for (91), we have

$$\begin{aligned}
\left\|M^{-\alpha}(g(t,q,p) - g(s,q,p))\right\|_{L^\infty(\mathbb{R}^2)} &\le \int_s^t \bigg[\big\|M^{-\alpha}\nabla\cdot(\mu)g\\
&\quad + M^{-\alpha}\mu\cdot\nabla g\big\|_{L^\infty(\mathbb{R}^2)} + \left\|M^{-\alpha}\frac{\sigma^2}{2}\frac{\partial^2}{\partial p^2}g\right\|_{L^\infty(\mathbb{R}^2)}\bigg]\mathrm{d}z\\
&\le \int_s^t \bigg[\left\|M^{1/2-2\alpha}M^{-1/2+\alpha}\left(\nabla\cdot(\mu)g + \mu\cdot\nabla g\right)\right\|_{L^\infty(\mathbb{R}^2)}\\
&\quad + \left\|M^{1/2-2\alpha}M^{-1/2+\alpha}\frac{\sigma^2}{2}\frac{\partial^2}{\partial p^2}g\right\|_{L^\infty(\mathbb{R}^2)}\bigg]\mathrm{d}z.
\end{aligned}\tag{94}$$

The terms $\|\nabla\cdot(\mu)M^{1/2-2\alpha}\|_{L^\infty(\mathbb{R}^2)}$, $\||\mu|M^{1/2-2\alpha}\|_{L^\infty(\mathbb{R}^2)}$ are bounded. We then use (89) and the Sobolev embedding theorem to deduce (91) from (94). The proof of (92) is analogous. ∎

PROPOSITION B.6. *Let $T > 0$. Let Assumption* (NG) *be satisfied. Let (q,p) obey the Langevin dynamics* (2). *Let $A(q,p) := p^{n_1}q^{n_2}$ for some $n_1, n_2 \in \mathbb{N}$, and let $c \ge 2$. Then, for any $s, t \in [0,T]$, we have*

$$\int_{\mathbb{R}} \left|\mathbb{E}\left[w_\epsilon(x-q(t))A(q(t),p(t)) - w_\epsilon(x-q(s))A(q(s),p(s))\right]\right|^c dx \leq C|t-s|^{1+\beta}, \tag{95}$$

$$\int_{\mathbb{R}} \left|\mathbb{E}\left[w'_\epsilon(x-q(t))A(q(t),p(t)) - w'_\epsilon(x-q(s))A(q(s),p(s))\right]\right|^c dx \leq C|t-s|^{1+\beta}, \tag{96}$$

where C is independent of $\epsilon > 0$. We also have for any $x \in \mathbb{R}$

$$\left|\mathbb{E}\left[w_\epsilon(x-q(t))A(q(t),p(t)) - w_\epsilon(x-q(s))A(q(s),p(s))\right]\right| \leq K|t-s|, \tag{97}$$

$$\left|\mathbb{E}\left[w'_\epsilon(x-q(t))A(q(t),p(t)) - w'_\epsilon(x-q(s))A(q(s),p(s))\right]\right| \leq K|t-s|, \tag{98}$$

where K is independent of $\epsilon > 0$ and $x \in \mathbb{R}$.

Proof. We rewrite the left-hand side of (95) as

$$\begin{aligned}&\int_{\mathbb{R}} \left|\mathbb{E}\left[w_\epsilon(x-q(t))A(q(t),p(t)) - w_\epsilon(x-q(s))A(q(s),p(s))\right]\right|^c \mathrm{d}x\\ &= \int_{\mathbb{R}} \left|\int_{\mathbb{R}}\int_{\mathbb{R}} w_\epsilon(x-q)A(q,p)(g(t,p,q)-g(s,p,q))\mathrm{d}p\mathrm{d}q\right|^c \mathrm{d}x = \|w_\epsilon * (\tilde{g}(\cdot,t)-\tilde{g}(\cdot,s))\|_c^c,\end{aligned}$$

where $\tilde{g}(q,t) := \int_{\mathbb{R}} A(q,p)g(t,q,p)\mathrm{d}p$. Let us define $h_{s,t}(q,p) := |(g(t,q,p)-g(s,q,p))|$. We proceed as

$$\|w_\epsilon * (\tilde{g}(\cdot,t)-\tilde{g}(\cdot,s))\|_c^c \leq \|w_\epsilon\|_1^c \|\tilde{g}(\cdot,t)-\tilde{g}(\cdot,s)\|_c^c \leq \int_{\mathbb{R}} \left|\int_{\mathbb{R}} |A(q,p)|\, h_{s,t}(q,p)\mathrm{d}p\right|^c \mathrm{d}q.$$

Fix $\theta \in (1/c, 2/c) \subset (0,1)$. We split $h_{s,t}(q,p) = h^{\theta}_{s,t}(q,p)h^{1-\theta}_{s,t}(q,p)$. We apply the Hölder inequality for this splitting in the above inner p-spatial integral, and we get

$$\begin{aligned}\int_{\mathbb{R}} \left|\int_{\mathbb{R}} A(q,p)h_{s,t}(q,p)\mathrm{d}p\right|^c \mathrm{d}q \quad &\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h_{s,t}(q,p)^2\mathrm{d}p\right)^{\theta c/2}\\ &\times \left(\int_{\mathbb{R}} |A(p,q)|^{\theta'} h_{s,t}(q,p)^{\theta''}\mathrm{d}p\right)^{c/\theta'} \mathrm{d}q,\end{aligned} \tag{99}$$

where $\theta'' := (1-\theta)\theta' > 0$, and θ' is conjugate to $2/\theta$. Let $\alpha \in (1/4, 1/2)$. We use (91) to deduce that

$$\begin{aligned}\int_{\mathbb{R}} |A(p,q)|^{\theta'} h_{s,t}(q,p)^{\theta''}\mathrm{d}p &= \int_{\mathbb{R}} |A(p,q)|^{\theta'} M^{\alpha\theta''}M^{-\alpha\theta''}h_{s,t}(q,p)^{\theta''}\mathrm{d}p\\ &\leq K\int_{\mathbb{R}} |A(p,q)|^{\theta'} M^{\alpha\theta''}\mathrm{d}p \leq K|q|^{n_2\theta'}\exp\{-CV(q)\}\end{aligned}$$

for some $C = C(n_1,\theta,\theta',\gamma,\sigma,\alpha) > 0$. We apply the Hölder inequality (in the q variable) in (99) to deduce

$$\begin{aligned}&\int_{\mathbb{R}} \left|\mathbb{E}\left[w_\epsilon(x-q(t))A(q(t),p(t)) - w_\epsilon(x-q(s))A(q(s),p(s))\right]\right|^c \mathrm{d}x\\ &\leq C\|h_{s,t}\|^{c\theta}_{L^2(\mathbb{R}^2)} \leq C|t-s|^{1+\beta},\end{aligned}$$

where we have used Lemma B.5, estimate (90), in the last inequality. We thus proved (95). The proof of (96) is similar. We can rewrite the left hand side of (96) as

$$\begin{aligned}&\int_{\mathbb{R}} \left|\int_{\mathbb{R}}\int_{\mathbb{R}} w'_\epsilon(x-q)A(q,p)(g(t,p,q)-g(s,p,q))\mathrm{d}p\mathrm{d}q\right|^c \mathrm{d}x\\ &= \int_{\mathbb{R}} \left|\int_{\mathbb{R}}\int_{\mathbb{R}} w_\epsilon(x-q)\frac{\partial}{\partial q}\left\{A(q,p)(g(t,p,q)-g(s,p,q))\right\}\mathrm{d}p\mathrm{d}q\right|^c \mathrm{d}x,\end{aligned} \tag{100}$$

where we have also used integration by parts in the q variable, and the fact that the integrands decay to 0 for $q \to \pm\infty$, by [15, Theorem 0.1]. From (100) onward, the computations carried out for (95) can now be adapted line by line with $\partial/\partial q \{A(q,p)g(t,q,p)\}$ replacing $A(q,p)g(t,q,p)$. This is possible because the q-derivative introduces a polynomial-type correction to $A(q,p)g(t,q,p)$, which can be dealt with as above, using again the exponential decay of M.

We turn to (97). We rely on (91) and compute

$$\begin{aligned}
&\left|\mathbb{E}\left[w_\epsilon(x-q(t))A(q(t),p(t)) - w_\epsilon(x-q(s))A(q(s),p(s))\right]\right| \\
&\quad\leq \int_{\mathbb{R}}\int_{\mathbb{R}} |w_\epsilon(x-q)A(q,p)(g(t,q,p)-g(s,q,p))| \,\mathrm{d}q\mathrm{d}p \\
&\quad\leq C|t-s| \int_{\mathbb{R}}\int_{\mathbb{R}} |w_\epsilon(x-q)A(q,p)M^\alpha| \,\mathrm{d}q\mathrm{d}p \\
&\quad\leq C|t-s| \int_{\mathbb{R}} \|w_\epsilon(x-\cdot)\|_{L^1}|p|^{n_1}\exp\{-C(\alpha,\gamma,\sigma)p^2/2\}\mathrm{d}p = K|t-s|,
\end{aligned}$$

which is the desired estimate. The proof of (98) is completely analogous, and it relies on integration by parts for w_ϵ' and estimate (92). ∎

Remark B.7. With the notation and assumptions of Proposition B.6, it is not difficult to adapt the proof of the same proposition to show that $\int_{\mathbb{R}} |\mathbb{E}[w_\epsilon(x-q(0))A(q(0)], p(0))|^c\,\mathrm{d}x$, $\int_{\mathbb{R}} |\mathbb{E}[w'_\epsilon(x-q(0))A(q(0),p(0))]|^c\,\mathrm{d}x$, $\int_{\mathbb{R}} |\mathbb{E}[w''_\epsilon(x-q(0))A(q(0),p(0))]|^c\,\mathrm{d}x$ are uniformly bounded in ϵ.

B.3. Estimate on negative powers of the density ρ_ϵ.

PROPOSITION B.8. *Assume the validity of either Assumption* (G) *or Assumption* (NG). *Let $N\epsilon^\theta = 1$ for some $\theta > 3$, and let ρ_ϵ be as in* (7). *Let $D \subset \mathbb{R}$ be a bounded set, and let $T > 0$ be fixed. As $N \to \infty$ and $\epsilon \to 0$, we have*

$$\mathbb{E}\left[\rho_\epsilon^{-2}(x,t)\right] \leq C(D,T) \qquad \text{for all } x \in D, \text{ for all } t \in [0,T], \tag{101}$$

where C is independent of N, ϵ.

Proof of Proposition B.8 *under Assumption* (G). We know that

$$q_i(t) \sim \mathcal{N}(\mu_q(t), \sigma_q^2(t)), \qquad t \in [0,T].$$

Also, $\mu_q(t)$ is bounded on $[0,T]$. We can think of the quantity $x - q_i(t)$ as being $(x-\mu_q(t))-(\mu_q(t)-q_i(t))$. This observation, together with the distributional symmetry of Gaussian random variables with mean zero, allows us to prove the statement by considering the simpler setting

$$\begin{aligned}
&q_i(t) \sim \mathcal{N}(0, \sigma_q^2(t)) \qquad \text{for all } t \in [0,T], \\
&0 \leq x \leq \max_{y\in D}|y| + \max_{s\in[0,T]}|\mu_q(s)| =: M,
\end{aligned}$$

without loss of generality. Notice that we have performed an abuse of notation with respect to q_i. We fix $t \in [0,T]$ and x satisfying the above condition. With our scaling choice $N = \epsilon^{-\theta}$, we have

$$\rho_\epsilon(x,t) = C\epsilon^{\theta-1}\sum_{i=1}^{N}\exp(-(q_i(t)-x)^2/2\epsilon^2).$$

For $\epsilon \leq 1$, there exists $\kappa = \kappa(D, T)$ such that

$$\kappa \cdot \epsilon \leq \underbrace{\mathbb{P}\left(q_i(t) \in (x-\epsilon, x+\epsilon)\right)}_{=:p_{x,t,\epsilon}} \qquad \text{for all } t \in [0,T], \text{ for all } x \in [0, M]. \tag{102}$$

A simple choice is $\kappa := (2/(2\pi\iota))\exp\{-(M+1)^2/2\nu\}$, where we have used Assumption (G).

The N particles being independent, we have

$$n(x,t) := \#\{\text{particles in } (x-\epsilon, x+\epsilon) \text{ at time } t\} \sim \text{ Bi}(N, p_{x,t,\epsilon}) = \text{Bi}(\epsilon^{-\theta}, p_{x,t,\epsilon}).$$

We fix a positive real number η. It then follows that, on the set $\{n(x,t) \geq 1\}$, we have

$$\frac{1}{\rho_\epsilon^\eta(x,t)} \leq \frac{1}{(n(x,t)\epsilon^{\theta-1})^\eta}.$$

Estimate on the set $\{n = 0\}$. We now focus on the set $\{n(x,t) = 0\}$. First of all, we notice that this event is asymptotically highly unlikely. More precisely, using the independence of particles, we get

$$\begin{aligned}\mathbb{P}\left(n(x,t) = 0\right) &= \mathbb{P}\left(\text{all particles in } (x-\epsilon, x+\epsilon)^C \text{ at time } t\right) = (1-p_{x,\epsilon,t})^N \\ &= (1-p_{x,\epsilon,t})^{\epsilon^{-\theta}} \leq (1-\kappa\epsilon)^{\epsilon^{-\theta}} \leq \exp\left\{-\epsilon^{-(\theta-1)}\frac{\kappa}{2}\right\}. \end{aligned} \tag{103}$$

Now that we have the asymptotic probability of finding no particles in $(x-\epsilon, x+\epsilon)$, we rely on the trivial bound $\rho_\epsilon(x,t) \geq w_\epsilon(x-\tilde{q}(t))N^{-1}$, where $\tilde{q}(t)$ is the closest particle to x at time t. In symbols, $\tilde{q}(t) := q_a(t)$, where $a := \arg\min_{i=1,\dots,N} |q_i(t) - x|$. We compute the probability density function for $|\tilde{q}(t) - x|$. For this purpose, we compute, for every $y \geq 0$,

$$\begin{aligned}\mathbb{P}\left(|x - \tilde{q}(t)| \leq y\right) &= 1 - \mathbb{P}\left(|\tilde{q}(t) - x| > y\right) \\ &= 1 - \mathbb{P}\left(\text{all particles in } (x-y; x+y)^C \text{ at time } t\right) \\ &= 1 - \mathbb{P}\left(q_1 \text{ in } (x-y; x+y)^C \text{ at time } t\right)^N \\ &= 1 - (\Phi_t(x-y) + 1 - \Phi_t(x+y))^N,\end{aligned}$$

where we have set $\Phi_t(z) := \int_{-\infty}^z \mathcal{G}(y, 0, \sigma_q^2(t))\mathrm{d}y$. In the rest of this proof only, we will shorten $\mathcal{G}(y, 0, \sigma_q^2(t))$ to simply $G_t(y)$. If we differentiate with respect to y, we get the probability density function for $|\tilde{q}(t) - x|$

$$f_{|\tilde{q}(t)-x|}(y) = \mathbf{1}_{y\geq 0} \cdot N(\underbrace{\Phi_t(x-y) + 1 - \Phi_t(x+y)}_{=:Z_{x,t}(y)})^{N-1}(G_t(x-y) + G_t(x+y)).$$

We now rely on the inequality

$$\mathbb{E}\left[\frac{1}{\rho_\epsilon^\eta(x,t)}\right] \leq \mathbb{E}\left[\frac{N^\eta}{w_\epsilon^\eta(\tilde{q}(t) - x)}\right].$$

We write the expectation on the right-hand side using the probability density function for $|\tilde{q} - x|$.

$$\mathbb{E}\left[\frac{N^\eta}{w_\epsilon^\eta(\tilde q(t)-x)}\right] = N^\eta \int_0^{+\infty} N(\Phi_t(x-y) + 1-\Phi_t(x+y))^{N-1}(G_t(x-y)+G_t(x+y))\frac{1}{w_\epsilon^2(y)}\mathrm{d}y. \tag{104}$$

Before we deal with (104), we need to estimate $Z_{x,t}(y)$, at least for large values of y. It is immediate to see that $Z_{x,t}(y) \le Z_{M,t}(y)$ for all $y \ge 0$. We compute the derivative

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{G}(z,0,\alpha) = C\exp\left\{-z^2/(2\alpha)\right\}\alpha^{-3/2}\left(z^2\alpha^{-1}-1\right).$$

Thanks to Assumption (G), this entails that

$$Z_{M,t}(y) \le Z_{M,\bar t}(y) \qquad \text{for } y \ge M+\sqrt{\iota}, \tag{105}$$

where we have set $\bar t := \arg\max_{s\in[0,T]}\sigma_q^2(s)$. We now examine the ratio $Z_{M,\bar t}(y)/G_{\bar t}(y-M)$. We use the l'Hôpital rule and compute

$$\begin{aligned}\lim_{y\to+\infty}\frac{Z_{M,\bar t}(y)}{G_{\bar t}(y-M)} &= \lim_{y\to+\infty}\frac{Z'_{M,\bar t}(y)}{G'_{\bar t}(y-M)} = \lim_{y\to+\infty}\frac{-G_{\bar t}(M-y)-G_{\bar t}(M+y)}{\frac{M-y}{\sigma_q^2(\bar t)}G_{\bar t}(y-M)}\\ &= \lim_{y\to+\infty}\left\{\frac{\sigma_q^2(\bar t)}{y-M}+\frac{\sigma_q^2(\bar t)}{y-M}\exp\left(-\frac{4My}{2\sigma_q^2(t)}\right)\right\} = 0.\end{aligned}$$

This implies the existence of $\overline{y} = \overline{y}(D,T) > M+\sqrt{\iota}$ such that

$$Z_{x,t}(y) \le Z_{M,\bar t}(y) \le \begin{cases}1 & \text{if } y \le \overline{y},\\ \exp\left(-\frac{(y-M)^2}{2\iota}\right) & \text{if } y \ge \overline{y}.\end{cases} \tag{106}$$

We are now able to compute (104) by splitting the integration on the two regions $[0,\overline{y}]$ and $[\overline{y},+\infty)$ provided by (106). We obtain

$$\begin{aligned}&\mathbb{E}\left[\frac{N^\eta}{w_\epsilon^\eta(\tilde q(t)-x)}\right]\\ &= N^\eta\int_0^{\overline{y}} N(\Phi_t(x-y)+1-\Phi_t(x+y))^{N-1}(G_t(x-y)+G_t(x+y))\frac{1}{w_\epsilon^\eta(y)}\mathrm{d}y\\ &\quad + N^\eta\int_{\overline{y}}^{+\infty} N(\Phi_t(x-y)+1-\Phi_t(x+y))^{N-1}(G_t(x-y)+G_t(x+y))\frac{1}{w_\epsilon^\eta(y)}\mathrm{d}y\\ &\le CN^\eta\underbrace{\int_0^{\overline{y}}\frac{N}{w_\epsilon^\eta(y)}\mathrm{d}y}_{=:T_1}\\ &\quad + N^\eta\underbrace{\int_{\overline{y}}^{+\infty} N\exp\left(-\frac{(y-M)^2(N-1)}{2\iota}\right)(G_t(x-y)+G_t(x+y))\frac{1}{w_\epsilon^\eta(y)}\mathrm{d}y}_{=:T_2}.\end{aligned}$$

Integral T_1 can be bounded as

$$\begin{aligned}\int_0^{\overline{y}}\frac{N}{w_\epsilon^\eta(y)}\mathrm{d}y &= CN\int_0^{\overline{y}}\epsilon^\eta\exp\left(\frac{\eta y^2}{2\epsilon^2}\right)\mathrm{d}y\\ &= C\epsilon^{\eta+1}N\int_0^{(\overline{y})/\epsilon} e^{z^2}\mathrm{d}z \le K_1(D,T,\eta)N\epsilon^\eta\exp\{K_2(D,T)\epsilon^{-2}\}.\end{aligned}$$

As for integral T_2, we notice that the scaling $N\epsilon^\theta = 1$ and the condition $\overline{y} > M + \sqrt{\iota}$ provide the bound

$$\frac{\eta y^2}{2\epsilon^2} - \frac{(y-M)^2(N-1)}{2\iota} \le -\frac{(y-M)^2}{4\iota/N} \qquad \text{for } N \ge \overline{N} = \overline{N}(D,T).$$

We can then estimate I_2 for $N \ge \overline{N}$, thus obtaining

$$\begin{aligned} I_2 &\le CN\epsilon^\eta \int_0^{+\infty} \exp\left\{\frac{\eta y^2}{2\epsilon^2} - \frac{(y-M)^2(N-1)}{2\iota}\right\} \mathrm{d}y \\ &\le CN\epsilon^\eta \int_0^{+\infty} \exp\left\{-\frac{(y-M)^2}{4\iota/N}\right\} \mathrm{d}y \le CN^{1/2}\epsilon^\eta. \end{aligned}$$

We combine the contributions of T_1 and T_2 and deduce

$$\mathbb{E}\left[\frac{N^\eta}{w_\epsilon^\eta(\tilde{q}(t)-x)}\right] \le K_1(D,T)N^\eta\epsilon^\eta\left\{N^{1/2} + N\exp(K_2(D,T)\epsilon^{-2})\right\}. \tag{107}$$

We set $\eta = 4$ and we deduce that

$$\begin{aligned} &\mathbb{E}\big[\rho_\epsilon^{-2}(x,t)\cdot \mathbf{1}_{\{n(x,t)=0\}}\big] \le \mathbb{E}\big[\rho_\epsilon^{-4}(x)\big]^{1/2}\mathbb{P}\left(n(x,t)=0\right)^{1/2} \\ &\qquad \le K_1(D,T)N^2\epsilon^2\left\{N^{1/2} + N\exp(K_2(D,T)\epsilon^{-2})\right\}^{1/2}\exp\left\{-\epsilon^{-(\theta-1)}\frac{\kappa}{4}\right\} \to 0, \end{aligned}$$

as $N \to \infty$ and $\epsilon \to 0$. The scaling $N\epsilon^\theta = 1$, with $\theta > 3$, is used to show the convergence to 0 of the above estimate. We have dealt with the expectation of $\rho_\epsilon^{-2}(x,t)$ on the set $\{n(x,t)=0\}$, uniformly over $x \in D$ and $t \in [0,T]$.

Estimate on the set $\{n \ge 1\}$. We now turn to the set $\{n(x,t) \ge 1\}$, and more precisely to estimating $\mathbb{E}\big[\rho_\epsilon^{-2}(x,t)\cdot \mathbf{1}_{\{n(x,t)\ge 1\}}\big]$. We have already noticed that on $\{n(x,t) \ge 1\}$ we have the bound

$$\frac{1}{\rho_\epsilon^2(x,t)} \le \frac{1}{(n(x,t)\epsilon^{\theta-1})^2}.$$

We use some tools from [6]. In particular, we estimate $\mathbb{E}[n(x,t)^{-2}]$ using [6, Corollary, section 2]. We have $\mathbb{E}[(n(x,t)+2)^{-2}] = \int_0^1 g_2(z)\mathrm{d}z$, where for $z \in [0,1]$

$$g_2(z) := z^{-1}\int_0^z g_1(u)\mathrm{d}u, \qquad g_1(z) := t(q+pz)^N,$$

and where we have abbreviated $p := p_{x,t,\epsilon}$, $q := 1 - p_{x,t,\epsilon}$. We bound g_2 as

$$\begin{aligned} g_2(z) &= z^{-1}\int_0^z u(q+pu)^N\mathrm{d}u \le \int_0^z (q+pu)^N\mathrm{d}u \\ &= p^{-1}\int_0^z \frac{\mathrm{d}}{\mathrm{d}u}\left\{\frac{(q+pu)^{N+1}}{N+1}\right\}\mathrm{d}u = \frac{(q+pz)^{N+1} - q^{N+1}}{p(N+1)}. \end{aligned}$$

We use the scaling $N = \epsilon^{-\theta}$ and proceed as

$$\begin{aligned} \mathbb{E}\big[(n(x,t)+2)^{-2}\big] = \int_0^1 g_2(u)\mathrm{d}u &\le \int_0^1 \frac{1}{p(N+1)}\left\{(q+pu)^{N+1} - q^{N+1}\right\}\mathrm{d}u \\ &\le \frac{q^{N+1}}{p(N+1)} + \frac{1}{p^2(N+1)(N+2)} \\ &\le \frac{\epsilon^{\theta-1}}{\kappa}\exp\left\{-\epsilon^{-(\theta-1)}\frac{\kappa}{2}\right\} + \frac{\epsilon^{2\theta-2}}{\kappa^2}. \end{aligned}$$

As a result we obtain

$$\begin{aligned}\mathbb{E}\left[\rho_\epsilon^{-2}(x,t)\cdot\mathbf{1}_{\{n(x,t)\ge 1\}}\right] &\le \mathbb{E}\left[\frac{1}{(n(x,t)\epsilon^{\theta-1})^2}\cdot\mathbf{1}_{\{n(x,t)\ge 1\}}\right]\\ &\le \frac{3^2}{\epsilon^{2\theta-2}}\mathbb{E}\left[\frac{1}{(n(x,t)+2)^2}\cdot\mathbf{1}_{\{n(x,t)\ge 1\}}\right]\\ &\le \frac{3^2}{\epsilon^{2\theta-2}}\mathbb{E}\left[\frac{1}{(n(x,t)+2)^2}\right] \le 3^2\left[\frac{\epsilon^{1-\theta}}{\kappa}\exp\left\{-\epsilon^{-(\theta-1)}\frac{\kappa}{2}\right\}+\frac{1}{\kappa^2}\right],\end{aligned}$$

which is uniformly bounded in ϵ, N. Combining the estimates on $\{n=0\}$ and $\{n\ge 1\}$ gives the result. ∎

Adaptation of the proof of Proposition B.8 *under Assumption* (NG). We need to check that (102) still holds and also adapt (106). The validity of (102) is a consequence of the theory of positive transition densities for degenerate diffusion stochastic differential equations; see [16, section 3] and [25].

Let us now consider $x\in D, t\in[0,T]$. We define $\Phi_t(z)$ to be the cumulative distribution function of $q_1(t)$. We need to estimate $Z_{x,t}(y) := \Phi_t(x-y)+1-\Phi_t(x+y)$ by providing a rapidly decaying estimate as $y\to+\infty$, similarly to (106). We use Lemma B.5 to deduce

$$f_{q(t)}(q) \le C\int_{\mathbb{R}} M^{1/2-\alpha}(q,p)\mathrm{d}p \le Ce^{-kV(q)}, \tag{108}$$

where $f_{q(t)}$ denotes the probability density function of $q_1(t)$, where $\alpha\in(1/4,1/2)$, where $k := (1/2-\alpha)(2\gamma/\sigma^2)$, and where M is given in Assumption (NG). For $y\ge 3\max_{x\in D}|x|$, we consider the limit

$$\lim_{y\to+\infty}\frac{Z_{x,t}(y)}{e^{-kV(y)}} \le \lim_{y\to+\infty}\frac{\int_{\mathbb{R}\setminus[-y/2;y/2]} e^{-kV(q)}\mathrm{d}q}{e^{-kV(y)}} \le C\lim_{y\to+\infty}\frac{-e^{-kV(y)}-e^{-kV(y)}}{V'(y)e^{-kV(y)}} = 0,$$

where we have used (108) is the first inequality, and the l'Hôpital rule and Assumption (NG) for the second inequality. The above limit, in combination with the growth rate of V (at least quadratic thanks to Assumption (NG)), guarantees that

$$Z_{x,t}(y) \le \begin{cases} 1 & \text{if } y\le\overline{y}=\overline{y}(D,T,V),\\ \exp\left(-\frac{y^2}{2\iota}\right) & \text{if } y\ge\overline{y}\end{cases}$$

for some $\iota>0$. The above estimate replaces (106) in the remaining part of the proof, which is unchanged. ∎

Remark B.9. The growth condition for V (i.e., the requirement $n\ge 1$, instead of $n>1/2$) is dictated by the adaptation of the proof of Proposition B.8. This stricter condition is not necessary for the proofs of Lemma B.5 and Proposition B.6.

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2.2. Conclusions

We have derived and analysed a regularised DK model based on stochastically independent Langevin particles, under two different sets of assumptions for the on-site potential V. As a key feature, our regularisation keeps track of particles' positions and momenta through a smooth kernel w_ϵ rather than through the (atomic) Dirac distribution function. The regularisation parameter ϵ is related to the number of particles through the scaling $N\epsilon^\theta = 1$, where θ is chosen large enough.

We started by writing down the evolution equation of relevant smoothed densities $\rho_\epsilon, j_\epsilon$ in (9). Equation (9) is not closed in $\rho_\epsilon, j_\epsilon$, due to the microscopic noise $\dot{\mathcal{Z}}_N$, and the auxiliary process $j_{2,\epsilon}$.

We proved several results, the first of which is Proposition 1.1. Here, we established tightness of all the components of (9) in the simultaneous limit of $N \to \infty$, $\epsilon \to 0$. The techniques deployed in the proof of this result, which are of independent interest, are the basis of the proofs of two subsequent results: in Proposition 1.2, uniqueness of the limit of ρ_ϵ (for $\epsilon \to 0$) was achieved; in Theorem 1.3, the microscopic noise $\dot{\mathcal{Z}}_N$ was replaced with the DK-type noise $\dot{\mathcal{Y}}_N$. The error associated with such replacement, which is negligible w.r.t. the noise size in the limit $N \to \infty$, $\epsilon \to 0$, is detailed in terms of the scaling parameter θ. The square-root feature of $\dot{\mathcal{Y}}_N$ is inherited from the stochastic independence of the particles' random driving forces, while the infinite-dimensional noise $\tilde{\xi}_\epsilon$ is of trace class because of the spatial smoothing entailed by the use of the kernel w_ϵ instead of Dirac deltas. Furthermore, we gave meaning to the conservative nature of the system by not combining the evolution equations for ρ_ϵ and j_ϵ, thus keeping the second-order in time structure of the model (as opposed to a first-order in time structure of (1.1)). In this way, the divergence operator acts on the stochastic noise only through the conservation of mass for ρ_ϵ (see (9), first equation), so there is no ambiguity as to the precise definition of the stochastic noise for the system. Combining these considerations with Proposition 1.2 and Theorem 1.3, we were able to justify the mesoscopic noise of (45b).

As for j_2, we approximated it with a multiple of $\partial\rho_\epsilon/\partial x$ under a low temperature assumption.

The overall result of the above modelling gave (45). After having been endowed with periodic boundary conditions, (45) becomes what we referred to as our regularised Dean-Kawasaki model (14). We then provided a suitable function setting in which we looked for mild solutions to (14). As (14) is a wave-type equation (thus, with no compact semigroup), we had to smooth the square-root singularity in the noise to build a mild solution. We relied on a small noise regime analysis and provided a uniqueness and existence result for a solution to (14) that stays bounded away from the zero (i.e., from the square-root singularity) in a high probability sense. This is the content of Theorem 1.6.

The points of strength of this work can be summarised as follows. Firstly, a rigorous derivation of a Dean-Kawasaki type model is made possible by the choice of a smooth function setting over the atomic setting. Secondly, a quantitative estimate on the 'cost' associated with the noise replacement (from microscopic to mesoscopic) is detailed. Thirdly, a proper definition of a conservative stochastic noise for the model is given by keeping a second-order structure (i.e., with no overdamped limit). Finally, the resulting model allows for smooth solutions (as opposed to (1.1)) in a high probability sense.

A number of questions remain open. Most importantly, the results we have produced do not give a solution defined with full probability. An associated criticality is the almost sure positivity for the density ρ_ϵ, which we can not achieve due to the fact the ρ_ϵ might significantly deviate (even if only with small probability) from the solution to the noise-free equivalent of (14). As a matter of fact, there is no component in (14) which prevents the solution from going negative. This is an indirect result of the chosen approximation of $j_{2,\epsilon}$, which leads to a stochastic perturbation of a wave equation.

The aspect of positivity of solutions for DK type equations appears to be crucial, and we will analyse it in more detail in Chapter 4. In Chapter 3, we extend the results obtained in this chapter to the important case of weakly interacting particles, thus allowing nonlocal interactions between the particles and, as a consequence, some form of stochastic dependence between them.

Chapter 3

From weakly interacting particles to a regularised Dean–Kawasaki model

In this chapter, we extend the contents of Chapter 2 to Langevin particle systems allowing weak nonlocal interactions via a pairwise potential. This is joint work with Tony Shardlow and Johannes Zimmer, and is available on arXiv [11].

3.1. Outline of the Article

As we have seen in the previous chapter, a regularisation of the mass-preserving noise of the DK equation (1.1) is derived from replacing the microscopic noise

$$\frac{\sigma}{N}\sum_{i=1}^{N} w_\epsilon(x-q_i(t))$$

with the mesoscopic noise

$$\frac{\sigma}{\sqrt{N}}\sqrt{\rho_\epsilon(x,t)}\tilde{\xi}_\epsilon,$$

with the Q-Wiener noise $\tilde{\xi}_\epsilon$ approximating a space-time white noise in the limit $\epsilon \to 0$. In particular, the specific nonlinear form of the mesoscopic noise is given by the independence of the Brownian motions driving the N particles, while it is not affected by stochastic independence of the particles themselves. In other words, while the derivation of the regularised DK model in the previous chapter benefits in many points from the independence of the particles, such independence is not necessary to derive the distinctive DK noise. As a matter of fact, the original DK equation (1.1) is derived from interacting particles [15]. It thus appears natural to adapt the regularisation arguments of the previous chapter to a system of interacting particles, so to describe more realistic and interesting cases. In particular, we consider particles weakly interacting via a pairwise potential W; these systems are intrinsically associated with a macroscopic nonlocal interaction term of type $\{W*\rho\}\,\rho$. On top of the arguments used in the previous chapter, we also rely on propagation of chaos techniques and Simon's compactness criterion. The propagation of chaos allows to compare, in the limit $N\to\infty$, our system of (dependent) particles to an auxiliary system of (independent) particles subject to McKean–Vlasov dynamics. The latter system can be dealt with using techniques from the previous chapter. The tightness analysis of relevant regularised quantities $\{\rho_\epsilon\}_\epsilon, \{j_\epsilon\}_\epsilon, \{j_{2,\epsilon}\}_\epsilon$ is dealt with using Simon's compactness criterion, rather the Kolmogorov's criterion. This addresses the lower time regularity entailed by the use of propagation of chaos.

We prove technical results (such as the propagation of chaos and relevant moment bounds) in Section 2. We provide the tightness analysis and relevant approximations (such as noise and drift replacements) and we obtain a regularised DK model for weakly interacting particles in Section 3. We analyse this model in Section 4.

Appendix B: Statement of Authorship

This declaration concerns the article entitled:									
From weakly interacting particles to a regularised Dean–Kawasaki model									
Publication status (tick one)									
draft manuscript		**Submitted**		**In review**	**X**	**Accepted**		**Published**	
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Candidate's contribution to the paper (detailed, and also given as a percentage).	The author of the thesis has performed the bulk of the computations for this work (70%). The presentation of the contents have been shared in equal weights between all authors (33%).								
Statement from Candidate	This paper reports on original research I conducted during the period of my Higher Degree by Research candidature.								
Signed		**Date**	17.9.2019						

From weakly interacting particles to a regularised Dean–Kawasaki model

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Abstract

The evolution of finitely many particles obeying Langevin dynamics is described by Dean–Kawasaki equations, a class of stochastic equations featuring a non-Lipschitz multiplicative noise in divergence form. We derive a regularised Dean–Kawasaki model based on second order Langevin dynamics by analysing a system of particles interacting via a pairwise potential. Key tools of our analysis are the propagation of chaos and Simon's compactness criterion. The model we obtain is a small-noise stochastic perturbation of the undamped McKean–Vlasov equation. We also provide a high-probability result for existence and uniqueness for our model.

Key words: Interacting particles, propagation of chaos, weakly self-consistent Vlasov-Fokker-Planck equation, Dean–Kawasaki model, mild solutions, second order Langevin dynamics.

AMS (MOS) Subject Classification: 60H15 (35R60)

1 Introduction

The Dean–Kawasaki model [6, 15] describes the evolution of a system of finitely many particles obeying Langevin dynamics. A key feature of the particle system is the stochastic independence of the forcing terms driving the particles. The particles themselves, on the other hand, might be independent [19] or interact through a potential [6]: in this work, we focus on the latter case.

In its simplest form, the Dean–Kawasaki model reads

$$\partial_t \rho = \nabla \cdot \left(\rho \nabla \frac{\delta F(\rho)}{\delta \rho}\right) + \nabla \cdot (\sigma \sqrt{\rho}\, \xi), \tag{1}$$

with $\sigma \in \mathbb{R}$, where ρ is the particle density, F is an energy functional, and ξ is a space-time white noise. The model (1) may be obtained from either a first-order Langevin equation [6], or from second-order Langevin dynamics in an overdamped limit [19].

Equations such as (1) pose a challenge for existence theory, in particular due to the multiplicative structure of the noise in divergence form and to its square-root coefficient function. The latter is related to the independence of the forcing terms driving the particles [6, 19]. Consequently, well-posedness for (1) is an open question, with the exception of the purely diffusive case [18]. More specifically, for the deterministic drift being $\frac{N}{2}\Delta$, where $N > 0$, equation (1) admits a unique trivial (atomic) solution only if $N \in \mathbb{N}$, and has no solutions if $N \notin \mathbb{N}$. This striking result indicates how subtle the analysis of equations of this kind is.

In order to obtain non-trivial solutions to (1), different approaches have been developed in recent years. One approach is to correct the drift [29, 2, 16, 17], another one is to regularise the equation [10, 21]. For a regularised undamped equivalent of (1), corresponding to a regularised stochastic wave equation in the density/momentum density pair (ρ, j), a result of existence and

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uniqueness is found in [4]; that model, here referred to as the *regularised Dean–Kawasaki* model, is derived from independent particles. The key regularisation chosen in [4] is a representation of particles by Gaussians, rather than their limiting Dirac measures. The main contributions of this work is to extend this idea to some important systems of interacting particles. Specifically, we derive and analyse a regularised Dean–Kawasaki model set in the undamped regime, as in [4], but describing the evolution of a system of finitely many weakly interacting particles governed by undamped McKean–Vlasov dynamics, see for example [9, 3, 24].

Throughout the paper, we rely on some methodology found in [4]. However, the interaction of the particles also requires various new approaches. Specifically, in contrast to [4], we employ propagation of chaos techniques [20] and Simon's compactness criterion [26] to overcome the difficulties posed by stochastically dependent particles. In addition, as the resulting model is superlinear (as specified below), we also need to localise the solutions using suitable stopping times. More details are provided in Subsection 1.2 below.

1.1 Weakly interacting particles on a one-dimensional torus

The system studied here consists of N interacting particles on the one-dimensional flat torus of length one, denoted by $\mathbb{T}$. Each particle $i \in \{1, \ldots, N\}$ is described in terms of position and velocity $(q_i, p_i) \in \mathbb{T} \times \mathbb{R}$. The system obeys the following undamped Langevin dynamics on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\begin{cases} \dot{q}_i = p_i, \\ \dot{p}_i = -\gamma p_i - \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} W'(q_i - q_j) + \sigma\,\dot{\beta}_i, \qquad i = 1, \ldots, N, \end{cases} \tag{2}$$

where $\{\beta_i\}_{i=1}^N$ are independent Brownian motions, the interaction potential W is periodic and smooth, say $W \in C^2(\mathbb{T})$, the initial conditions $\{(q_{i,0}, p_{i,0})\}_{i=1}^N$ are independent and identically distributed, and σ and γ are positive constants. The dissipative term $-\gamma p_i$ is a frictional drag, balancing the fluctuating Brownian term $\sigma\dot{\beta}_i$. The particles $\{(q_i, p_i)\}_{i=1}^N$ are exchangeable, but not necessarily independent.

Remark 1.1. Throughout this work, diacritical dots ($\dot{}$) are used to indicate time differentiation of finite or infinite dimensional Itô processes (e.g., see (2)).

In order to study (2), we introduce an auxiliary Langevin system of particles $\{(\overline{q}_i, \overline{p}_i)\}_{i=1}^N$ obeying

$$\begin{cases} \dot{\overline{q}}_i = \overline{p}_i, \\ \dot{\overline{p}}_i = -\gamma \overline{p}_i - W' * \mu_t(\overline{q}_i) + \sigma\,\dot{\beta}_i, \qquad i = 1, \ldots, N, \end{cases} \tag{3}$$

where $*$ denotes the convolution operator on $\mathbb{T}$, μ_t denotes the law of $\overline{q}_i(t)$, and the Brownian motions and the initial conditions coincide $\mathbb{P}$-a.s. with their respective counterparts in (2). As a result of these assumptions, the particles $\{(\overline{q}_i, \overline{p}_i)\}_{i=1}^N$ are clearly independent. System (3) is associated with the Vlasov–Fokker–Planck equation

$$\frac{\partial f_t}{\partial t} + p\frac{\partial f_t}{\partial q} - W' * \rho[f_t](q)\frac{\partial f_t}{\partial p} = \frac{\sigma^2}{2}\Delta_p f_t + \frac{\partial(\gamma p f_t)}{\partial p} \tag{4}$$

in the probability density function $f_t(q,p)\colon [0,T] \times \mathbb{T} \times \mathbb{R} \to [0, \infty)$, where $\rho[f_t](q) = \int_{\mathbb{R}} f_t(q,p)\mathrm{d}p$; see [3, 28].

1.2 Outline of the paper

We derive and analyse a regularised Dean–Kawasaki model in the undamped regime, based on the interacting particle system (2). A portion of our analysis is based on [4], and the relevant methodological novelties are sketched and put into context below.

Section 2 contains some auxiliary results. Subsection 2.1 establishes a propagation of chaos result (Proposition 2.1) linking (2) and (3), using ideas from [22, 20]. This sort of result, which is not required in [4], is here needed to compare the system of interest (2) to the more tractable system of independent particles (3). Specific aspects of the latter system's regularity, and in particular of the regularity of solutions to (4), are studied in Proposition 2.3 in Subsection 2.2; there, we explain the reason for choosing $\mathbb{T}$ (rather than $\mathbb{R}$ as in [4]) as the spatial domain. Subsection 2.3 relies on Propositions 2.1 and 2.3 to establish Proposition 2.6: for $\epsilon > 0$, this result provides ϵ-independent uniform estimates for certain Sobolev-space norms applied to the *regularised* densities

$$\rho_\epsilon(x,t) := \frac{1}{N}\sum_{i=1}^{N} w_\epsilon(x-q_i(t)), \quad j_\epsilon(x,t) := \frac{1}{N}\sum_{i=1}^{N} p_i(t) w_\epsilon(x-q_i(t)), \tag{5}$$

$$j_{2,\epsilon}(x,t) := \frac{1}{N}\sum_{i=1}^{N} p_i^2(t) w'_\epsilon(x-q_i(t)). \tag{6}$$

Above, $(x,t) \in \mathbb{T}\times[0,T]$, while w_ϵ is the periodic von Mises distribution [12] on $\mathbb{T}$ with location parameter $\mu := 0$ and concentration parameter $\kappa := \epsilon^{-2}$, namely,

$$w_\epsilon(x) := Z_\epsilon^{-1} e^{-\frac{\sin^2(x/2)}{\epsilon^2/2}}, \qquad Z_\epsilon := \int_{\mathbb{T}} e^{-\frac{\sin^2(x/2)}{\epsilon^2/2}} \mathrm{d}x. \tag{7}$$

The quantities in (5) are the *regularised* empirical density and momentum density for (2), and will be the building block of our final model; as for (6), this is a relevant auxiliary quantity emerging from the analysis of (5).

The kernel w_ϵ is introduced for smoothing and regularisation purposes. More precisely, we work with the quantities (5)–(6) rather than their atomic counterparts defined by a replacement of w_ϵ with Dirac delta functions centred on the particles; this is a key aspect of our approach, as it allows us to use standard tools from stochastic analysis and work with smooth functions. We refer to [4, Section 1] for a similar discussion. The kernel w_ϵ, which recovers a Dirac delta as $\epsilon \to 0$, is the toroidal equivalent of a Gaussian distribution with variance ϵ^2. The basic inequality $|x/4| \le |\sin(x/2)| \le |x/2|$, valid for all $x \in [0,\pi]$, implies that the ϵ-scalings of all the moments of w_ϵ are identical to those of a Gaussian of variance ϵ^2. In particular, we have that $C_1\epsilon \le Z_\epsilon \le C_2\epsilon$, for some constants $C_2 > C_1 > 0$. We can thus effectively use the kernel w_ϵ as if it is a Gaussian of variance ϵ^2, thus reusing much of scaling considerations (of polynomial type in ϵ^{-1} and N^{-1}) found in [4], where w_ϵ is Gaussian.

Remark 1.2. Throughout the paper, the quantities in (5)–(6) will always be understood under scalings of the type $N\epsilon^\theta = 1$, for θ large enough. Such a scaling is convenient to deal with the simultaneous limits $\epsilon \to 0$ and $N \to \infty$. This is because most bounds that we will prove with respect to (5)–(6) feature a polynomial contribution in ϵ^{-1} and N^{-1}, as mentioned above.

Section 3 is concerned with the evolution of the particle system (2). Subsection 3.1 contains Proposition 3.2, which provides relative compactness in law for the families $\{\rho_\epsilon\}_\epsilon$, $\{j_\epsilon\}_\epsilon$, and $\{j_{2,\epsilon}\}_\epsilon$ in the limit $\epsilon \to 0$. In this result, the crucial feature of time regularity of the processes is settled not by the Kolmogorov criterion [14, Corollary 14.9] (as for the corresponding result in [4]), but by

Simon's compactness criterion [26, Theorem 5] applied in the context of the Prokhorov Theorem [14]. The need for the latter method arises since the estimates for the time regularity obtained here are less sharp than those in [4], due to the use of the propagation of chaos (Proposition 2.1).

We then focus on the evolution equations for (5), which are the building blocks of our regularised Dean-Kawasaki model. As the evolution equations for (5) are not closable in (5), we rely on three relevant approximations. The first one, explained in Subsection 3.2, provides the distinctive particle interaction term $\{W' * \rho_\epsilon\}\,\rho_\epsilon$. The second one, detailed in Subsection 3.3, gives the relevant Dean–Kawasaki type noise (depending on ρ_ϵ and on a regular infinite-dimensional noise). The key differences with respect to the analogous argument performed in [4] (these being primarily due to the use of the propagation of chaos, the use of the von Mises kernels, and the lack of control over inverse powers of ρ_ϵ in the case of dependent particles) are explained there. The third and final approximation, which we justify in a low-temperature regime, allows us to replace $j_{2,\epsilon}$ (defined in (6)) with a multiple of $\partial\rho_\epsilon/\partial x$.

In Section 4 we take advantage of the approximations discussed above and derive our *regularised Dean-Kawasaki* model for weakly interacting particles in undamped regime

$$\begin{cases} \dfrac{\partial\tilde\rho_\epsilon}{\partial t}(x,t) = -\dfrac{\partial\tilde j_\epsilon}{\partial x}(x,t), & \text{(8a)}\\[2mm] \dfrac{\partial\tilde j_\epsilon}{\partial t}(x,t) = -\gamma\tilde j_\epsilon(x,t) - \left(\dfrac{\sigma^2}{2\gamma}\right)\dfrac{\partial\tilde\rho_\epsilon}{\partial x}(x,t) - \{W' * \tilde\rho_\epsilon(\cdot,t)\}(x)\tilde\rho_\epsilon(x,t) + \dfrac{\sigma}{\sqrt N}\sqrt{\tilde\rho_\epsilon(x,t)}\,\tilde\xi_\epsilon, & \text{(8b)}\\[2mm] \tilde\rho_\epsilon(x,0) = \rho_0(x), \quad \tilde j_\epsilon(x,0) = j_0(x), \end{cases}$$

for $(x,t) \in \mathbb{T}\times[0,T]$, where (ρ_0, j_0) is a suitable initial datum, where $\tilde\xi_\epsilon$ is a regular Q-Wiener process (e.g., in the sense of [25]), and where the aforementioned approximations are visible in the last three terms of the right-hand side of (8b). We use $(\tilde\rho_\epsilon, \tilde j_\epsilon)$ to refer to the solution of the approximate model (8), and $(\rho_\epsilon, j_\epsilon)$ to refer to the original densities in (5).

We provide a few preliminary results concerning the existence of local mild solutions to (8) and also to its noise-free version. We then prove the main existence and uniqueness result of the paper, Theorem 4.4. More specifically, we perform a small-noise regime analysis, in a similar way to the one carried out in [4], to prove a high-probability existence and uniqueness result of mild solutions to (8). On top of the arguments in [4], additional localisation procedures via stopping times and the conservation of mass for the system are needed to treat the locally bounded (superlinear) interaction term $\{W' * \tilde\rho_\epsilon\}\tilde\rho_\epsilon$.

2 Preliminary results

We prove a few results which will be used in Section 3 for the derivation of the undamped regularised Dean–Kawasaki model for weakly interacting particles.

2.1 Propagation of chaos

We first quantify how much the particles in (2) follow their counterparts in (3).

Proposition 2.1 (Propagation of chaos)**.** *Let $N \in \mathbb{N}$, let $\alpha \geq 2$ be an even natural number, let $T > 0$, and let $W \in C^2(\mathbb{T})$. There exists a constant $C = C(W,T,\alpha)$ such that*

$$\sup_{t\in[0,T]} \mathbb{E}\big[|q_1(t) - \overline{q}_1(t)|^\alpha + |p_1(t) - \overline{p}_1(t)|^\alpha\big]^{\frac{1}{\alpha}} \leq \frac{C(W,T,\alpha)}{\sqrt N}, \tag{9}$$

where the particle notation is inherited from (2) *and* (3)*.*

Proof. We adapt the proof of [20, Theorem 3.3]. Let $\beta_N(t) := \mathbb{E}[|q_1(t) - \overline{q}_1(t)|^\alpha + |p_1(t) - \overline{p}_1(t)|^\alpha]$. We apply the Itô formula for the function $f(z) = |z|^\alpha$ applied to the processes $q_i(t) - \overline{q}_i(t)$ and $p_i(t) - \overline{p}_i(t)$, for each $i \in \{1, \dots, N\}$, and sum the results. We notice that the stochastic noise for $p_i(t) - \overline{p}_i(t)$, $i \in \{1, \dots, N\}$, vanishes by assumption. We obtain

$$\sum_{i=1}^N |q_i(t) - \overline{q}_i(t)|^\alpha = \int_0^t \sum_{i=1}^N \alpha (q_i(r) - \overline{q}_i(r))^{\alpha-1} (p_i(r) - \overline{p}_i(r)) \mathrm{d}r =: T_1, \tag{10a}$$

$$\begin{aligned} \sum_{i=1}^N |p_i(t) - \overline{p}_i(t)|^\alpha &= -\frac{\alpha}{N} \int_0^t \sum_{i,j=1}^N (p_i(r) - \overline{p}_i(r))^{\alpha-1} \left(W'(q_i(r) - q_j(r)) - W' * \mu_r(\overline{q}_i(r))\right) \mathrm{d}r \\ &\quad + \int_0^t \sum_{i=1}^N \alpha (p_i(r) - \overline{p}_i(r))^{\alpha-1} (-\gamma \left[p_i(r) - \overline{p}_i(r)\right]) \mathrm{d}r =: T_2 + T_3. \end{aligned} \tag{10b}$$

We bound T_1 using the Young inequality with exponents α and $\alpha/(\alpha - 1)$. We thus obtain for $T_1 + T_3$

$$T_1 + T_3 \leq C(\alpha, \gamma) \int_0^t \sum_{i=1}^N (|q_i(r) - \overline{q}_i(r)|^\alpha + |p_i(r) - \overline{p}_i(r)|^\alpha) \mathrm{d}r. \tag{11}$$

As for T_2, we rewrite it as $T_2 = -\frac{\alpha}{N} \int_0^t \sum_{i,j=1}^N \left\{c_{ij}^{(1)}(r) + c_{ij}^{(2)}(r)\right\} \mathrm{d}r$, where

$$\begin{aligned} c_{ij}^{(1)}(r) &:= \left[W'(q_i(r) - q_j(r)) - W'(\overline{q}_i(r) - \overline{q}_j(r))\right] (p_i(r) - \overline{p}_i(r))^{\alpha-1}, \\ c_{ij}^{(2)}(r) &:= \left[W'(\overline{q}_i(r) - \overline{q}_j(r)) - W' * \mu_r(\overline{q}_i(r))\right] (p_i(r) - \overline{p}_i(r))^{\alpha-1}. \end{aligned}$$

We use the boundedness of W'', a Taylor expansion of W', and the Young inequality with exponents α and $\alpha/(\alpha - 1)$ to find

$$\begin{aligned} &\left| -\frac{\alpha}{N} \int_0^t \sum_{i,j=1}^N c_{ij}^{(1)}(r) \mathrm{d}r \right| \\ &\leq \frac{\alpha}{N} \int_0^t \sum_{i,j=1}^N \left|W'(q_i(r) - q_j(r)) - W'(\overline{q}_i(r) - \overline{q}_j(r))\right| |p_i(r) - \overline{p}_i(r)|^{\alpha-1} \mathrm{d}r \\ &\leq \frac{C(W,\alpha)}{N} \int_0^t \sum_{i,j=1}^N \left\{|q_i(r) - \overline{q}_i(r)| + |q_j(r) - \overline{q}_j(r)|\right\} |p_i(r) - \overline{p}_i(r)|^{\alpha-1} \mathrm{d}r \\ &\leq \frac{C(W,\alpha)}{N} \int_0^t \sum_{i,j=1}^N \{|q_i(r) - \overline{q}_i(r)|^\alpha + |q_j(r) - \overline{q}_j(r)|^\alpha + |p_i(r) - \overline{p}_i(r)|^\alpha\} \mathrm{d}r \\ &= C(W,\alpha) \int_0^t \sum_{i=1}^N \{|q_i(r) - \overline{q}_i(r)|^\alpha + |p_i(r) - \overline{p}_i(r)|^\alpha\} \mathrm{d}r. \end{aligned} \tag{12}$$

Fix $r \in [0, t]$ and $i \in \{1, \dots, N\}$. We employ the Hölder inequality with exponents α and $\alpha/(\alpha - 1)$

to obtain

$$\mathbb{E}\left[\sum_{j=1}^{N} c_{ij}^{(2)}(r)\right] = \mathbb{E}\left[\sum_{j=1}^{N} \left[W'(\overline{q}_i(r)-\overline{q}_j(r)) - W' * \mu_r(\overline{q}_i(r))\right] (p_i(r)-\overline{p}_i(r))^{\alpha-1}\right]$$
$$\leq \mathbb{E}\big[|p_i(r)-\overline{p}_i(r)|^{\alpha}\big]^{(\alpha-1)/\alpha}\theta_i^{1/\alpha}(r), \tag{13}$$

where

$$\theta_i(r) := \mathbb{E}\left[\left|\sum_{j=1}^{N} \xi_{\overline{q}_i(r),\overline{q}_j(r)}\right|^{\alpha}\right] = \mathbb{E}\left[\left(\sum_{j=1}^{N} \xi_{\overline{q}_i(r),\overline{q}_j(r)}\right)^{\alpha}\right],$$

with $\xi_{\overline{q}_i(r),\overline{q}_j(r)} := W'(\overline{q}_i(r)-\overline{q}_j(r)) - W' * \mu_r(\overline{q}_i(r))$, and where we have also used the fact that α is an even natural number. We define

$$\mathcal{T}_{1,\alpha} := \left\{\mathbf{j} = (j_1,\dots,j_\alpha) \in \{1,\dots,N\}^\alpha : \exists j_k \neq i \text{ such that } j_k \text{ appears exactly once in } \mathbf{j}\right\},$$
$$\mathcal{T}_{2,\alpha} := \left\{\mathbf{j} = (j_1,\dots,j_\alpha) \in \{1,\dots,N\}^\alpha : \mathbf{j} \notin \mathcal{T}_{1,\alpha}\right\}.$$

We have $\#\mathcal{T}_{2,\alpha} \leq C(\alpha)N^{\alpha/2}$, where $\#$ denotes set cardinality. To see this, consider a generic $\mathbf{j} \in \mathcal{T}_{2,\alpha}$. There are at most $\alpha/2$ values attained in $\mathbf{j}$: arguing by contradiction, if this is not the case, then i is attained exactly once (due to the definition of $\mathcal{T}_{2,\alpha}$). However, this means that the remaining $\alpha-1$ occurrences of $\mathbf{j}$ are distributed among at least $\alpha/2$ values, granting the existence of $j_k \neq i$ appearing exactly once in $\mathbf{j}$, and thus contradicting the definition of $\mathcal{T}_{2,\alpha}$. We therefore have no more than $C(\alpha)N^{\alpha/2}$ possible configurations in $\mathcal{T}_{2,\alpha}$, where $C(\alpha)$ is a suitable constant. We expand the definition of $\theta_i(r)$ as

$$\theta_i(r) = \sum_{\mathbf{j}\in\mathcal{T}_{1,\alpha}} \mathbb{E}\left[\prod_{k=1}^{\alpha} \xi_{\overline{q}_i(r),\overline{q}_{j_k}(r)}\right] + \sum_{\mathbf{j}\in\mathcal{T}_{2,\alpha}} \mathbb{E}\left[\prod_{k=1}^{\alpha} \xi_{\overline{q}_i(r),\overline{q}_{j_k}(r)}\right].$$

For any $\mathbf{j} \in \mathcal{S}_{1,\alpha}$, it holds that $\mathbb{E}\Big[\prod_{k=1}^{\alpha} \xi_{\overline{q}_i(r),\overline{q}_{j_k}(r)}\Big] = 0$. To see this, let $z \in \mathbb{T}$, and let $j \neq i$ be an index appearing just once in $\mathbf{j}$. Then

$$\mathbb{E}\left[\prod_{k=1}^{\alpha} \xi_{\overline{q}_i(r),\overline{q}_{j_k}(r)}\middle|\overline{q}_i(r)=z\right] = \prod_{j_k=i}\xi_{z,z}\cdot\mathbb{E}\left[\left(\prod_{j_k\neq i, j_k\neq j} \xi_{z,\overline{q}_{j_k}(r)}\right)\xi_{z,\overline{q}_j(r)}\middle|\overline{q}_i(r)=z\right]$$
$$= \prod_{j_k=i}\xi_{z,z}\cdot\mathbb{E}\left[\left(\prod_{j_k\neq i, j_k\neq j} \xi_{z,\overline{q}_{j_k}(r)}\right)\xi_{z,\overline{q}_j(r)}\right] \tag{14}$$
$$= \prod_{j_k=i}\xi_{z,z}\cdot\mathbb{E}\left[\prod_{j_k\neq i, j_k\neq j} \xi_{z,\overline{q}_{j_k}(r)}\right]\mathbb{E}\Big[\xi_{z,\overline{q}_j(r)}\Big] \tag{15}$$
$$= \prod_{j_k=i}\xi_{z,z}\cdot\mathbb{E}\left[\prod_{j_k\neq i, j_k\neq j} \xi_{z,\overline{q}_{j_k}(r)}\right]\mathbb{E}\big[\big(W'(z-\overline{q}_j(r)) - W' * \mu_r(z)\big)\big] = 0, \tag{16}$$

where independence of particles is used in (14) and (15), and $\mathbb{E}\big[\big(W'(z-\overline{q}_j(r)) - W' * \mu_r(z)\big)\big] = 0$ settles (16). The exchangeability of particles, the Hölder inequality, the boundedness of W', and the bound $\#\mathcal{T}_{2,\alpha} \leq C(\alpha)N^{\alpha/2}$ then give

$$\begin{aligned}\theta_i(r) &= \sum_{\mathbf{j}\in\mathcal{S}_{2,\alpha}} \mathbb{E}\left[\prod_{k=1}^{\alpha} \xi_{\overline{q}_i(r),\overline{q}_{j_k}(r)}\right] \\ &\leq C(\alpha)N^{\frac{\alpha}{2}}\mathbb{E}\left[|W'(\overline{q}_1(r)-\overline{q}_2(r))|^{\alpha} + |W' * \mu_r(\overline{q}_1(r))|^{\alpha}\right] \leq C(W,\alpha)N^{\frac{\alpha}{2}}.\end{aligned} \tag{17}$$

We sum (10a) and (10b), combine (11), (12), (13), and (17), and use the exchangeability of the particles to obtain

$$\beta_N(t) \leq \int_0^t C(\alpha,\gamma)\beta_N(r)\mathrm{d}r + \int_0^t C(W,\alpha)N^{-1/2}(\beta_N(r))^{(\alpha-1)/\alpha}\mathrm{d}r. \tag{18}$$

Applying the Young inequality in the second integral of (18) and then Gronwall's inequality completes the proof. □

We point out a couple of differences between Proposition 2.1 and [20, Theorem 3.3]. Firstly, we do not require convexity for the interaction potential W, as we are only interested in an estimate up to a given finite time; there is thus no need for a dissipative term in (18). Secondly, since the derivative W' is bounded, we can choose α arbitrarily large without violating the validity of (17). In the proof of Proposition 2.6 below, we will pick $\alpha > 2$.

2.2 Fokker–Planck regularity estimates

We now establish useful regularity properties of the particle system (3). We use C^n to denote n times continuously differentiable functions on $\mathbb{T}$, for $n \in \mathbb{N}\cup\{0\}$. We first specify our assumptions on (3).

Assumption 2.2. We assume that the initial datum $(\overline{q}(0),\overline{p}(0))$ of (3) coincides with $(\overline{q}_{\mathrm{aux}}(t_0),\overline{p}_{\mathrm{aux}}(t_0))$ for some $t_0 > 0$, where $(\overline{q}_{\mathrm{aux}},\overline{p}_{\mathrm{aux}})$ is an auxiliary process satisfying (3) and starting from an initial datum distributed according to a probability density f_0 satisfying

$$\int_{\mathbb{T}}\int_{\mathbb{R}} f_0(q,p)(1+p^2)^k\, dp\, dq < \infty.$$

Our choice to only consider a process "restarted" at some time $t_0 > 0$ is motivated by the need of the uniform-in-time Sobolev estimates found in [28, (17.2)], which we will use in the following result.

Proposition 2.3. *For $n, n_1 \in \mathbb{N}\cup\{0\}$ and $c \geq 2$, let w be a C^n-probability density function and $g \in C^n$. Let the initial datum of* (3) *be as specified in Assumption 2.2. Then*

$$\int_{\mathbb{T}}\left|\mathbb{E}\left[g(\overline{q}(t))\overline{p}^{n_1}(t)\frac{\partial^n}{\partial x^n}w(x-\overline{q}(t))\right]\right|^c \mathrm{d}x \leq C(g,t_0,f_0,n), \qquad \textit{for all } t \geq 0,$$

where $C(g,t_0,f_0,n)$ does not depend on w.

Proof. We first prove that, for $f_t(q,p)$ being the probability density function of $(\overline{q}(t),\overline{p}(t))$ and for any $\tilde{g} \in C^0$, we have

$$\int_{\mathbb{T}}\left|\int_{\mathbb{R}}|\tilde{g}(q)p^{n_1}|\left|\frac{\partial^m}{\partial q^m}f_t(q,p)\right|\mathrm{d}p\right|^c \mathrm{d}q \leq C(\tilde{g},t_0,f_0,n), \qquad \text{for } m \in \{0,1,\dots,n\}. \tag{19}$$

We use the boundedness of g and the Hölder inequality with exponents c and $c/(c-1)$ to obtain

$$\begin{aligned}
&\int_{\mathbb{T}}\left|\int_{\mathbb{R}}|\tilde g(q)p^{n_1}|\left|\frac{\partial^m}{\partial q^m}f_t(q,p)\right|\mathrm{d}p\right|^c\mathrm{d}q\le C(\tilde g)\int_{\mathbb{T}}\left|\int_{\mathbb{R}}|p^{n_1}|\left|\frac{\partial^m}{\partial q^m}f_t(q,p)\right|^{\frac{2}{c}}\left|\frac{\partial^m}{\partial q^m}f_t(q,p)\right|^{\frac{c-2}{c}}\mathrm{d}p\right|^c\mathrm{d}q\\
&\le C(\tilde g)\int_{\mathbb{T}}\left(\int_{\mathbb{R}}|p^{n_1c}|\left|\frac{\partial^m}{\partial q^m}f_t(q,p)\right|^2|1+p^2|^{kc}\,\mathrm{d}p\right)\left(\int_{\mathbb{R}}\left|\frac{\partial^m}{\partial q^m}f_t(q,p)\right|^{\frac{c-2}{c-1}}|1+p^2|^{-\frac{kc}{c-1}}\,\mathrm{d}p\right)^{c-1}\mathrm{d}q.
\end{aligned}\tag{20}$$

The second p-integral in (20) can be bounded by a constant $C(t_0,f_0,n)$, provided we pick $k>\frac{c-1}{2c}$. To see this, we notice that [28, (17.2)] gives uniform bounds in time for $\|f_t\|_{W^{n+2,2}(\mathbb{T}\times\mathbb{R})}$, where we use the Sobolev space notation. The continuous embedding $W^{n+2,2}(\mathbb{T}\times\mathbb{R})\subset C^m(\mathbb{T}\times\mathbb{R})$, which is a result of the application of [1, Theorem 4.12, Part I, Case A, equation (1)]) thus implies that

$$\sup_{q\in\mathbb{T},p\in\mathbb{R}}\left|\frac{\partial^m}{\partial q^m}f_t(q,p)\right|\le C(t_0,f_0,n),\qquad\text{for all } t\ge 0.$$

As a result, the argument of the second p-integral in (20) is controlled by $(1+p^2)^{-\frac{kc}{c-1}}$, which is integrable thanks to the choice of k. Thus (20) is bounded by

$$C(\tilde g,t_0,f_0,n)\int_{\mathbb{T}}\int_{\mathbb{R}}|p^{n_1c}|\left|\frac{\partial^m}{\partial q^m}f_t(q,p)\right|^2|1+p^2|^{kc}\,\mathrm{d}p\mathrm{d}q,$$

which is in turn uniformly bounded in time due to [28, (17.2)]. We have thus verified (19). We now define $\tilde f_t(q):=\int_{\mathbb{R}}(\partial^n/\partial q^n)\{g(q)p^{n_1}f_t(q,p)\}\,\mathrm{d}p$. We use integration by parts and Young's inequality for convolutions to bound

$$\begin{aligned}
&\int_{\mathbb{T}}\left|\mathbb{E}\left[g(\overline{q}(t))\overline{p}^{n_1}(t)\frac{\partial^n}{\partial x^n}w(x-\overline{q}(t))\right]\right|^c\mathrm{d}x=\int_{\mathbb{T}}\left|\int_{\mathbb{T}}\int_{\mathbb{R}}g(q)p^{n_1}f_t(q,p)\frac{\partial^n}{\partial q^n}w(x-q)\mathrm{d}p\mathrm{d}q\right|^c\mathrm{d}x\\
&=\int_{\mathbb{T}}\left|\int_{\mathbb{T}}\int_{\mathbb{R}}w(x-q)\frac{\partial^n}{\partial q^n}\{g(q)p^{n_1}f_t(q,p)\}\,\mathrm{d}p\mathrm{d}q\right|^c\mathrm{d}x\\
&=\int_{\mathbb{T}}\left|\int_{\mathbb{T}}w(x-q)\tilde f_t(q)\mathrm{d}q\right|^c\mathrm{d}x=\|w*\tilde f_t\|^c_{L^c(\mathbb{T})}\le\|w\|^c_{L^1(\mathbb{T})}\|\tilde f_t\|^c_{L^c(\mathbb{T})}=\|\tilde f_t\|^c_{L^c(\mathbb{T})}\\
&=\int_{\mathbb{T}}\left|\int_{\mathbb{R}}\frac{\partial^n}{\partial q^n}\{g(q)p^{n_1}f_t(q,p)\}\,\mathrm{d}p\right|^c\mathrm{d}q\\
&\le C(n,c)\sum_{j=0}^n\int_{\mathbb{T}}\left|\int_{\mathbb{R}}\left|\frac{\partial^j}{\partial q^j}\{g(q)\}\,p^{n_1}\frac{\partial^{n-j}}{\partial q^{n-j}}\{f_t(q,p)\}\right|\mathrm{d}p\right|^c\mathrm{d}q.
\end{aligned}\tag{21}$$

As $g\in C^n$, it is clear that each of the $(n+1)$ terms in (21) is as prescribed by the left-hand-side of (19), for some appropriate choices of $\tilde g$ and m. The proof is complete. □

Remark 2.4. The use of [28, (17.2)] is the reason for having $\mathbb{T}$, and not $\mathbb{R}$, as the spatial domain.

Remark 2.5. With the same notation and assumptions of Propositions 2.1 and 2.3, let the initial datum of the particles systems (2) and (3) have density $(\overline{q}_{\mathrm{aux}}(t_0),\overline{p}_{\mathrm{aux}}(t_0))$. It is easy to prove that the particle systems (2) and (3) have moments of any order uniformly bounded on $[0,T]$. This is a simple consequence of the boundedness of W'.

2.3 A useful application of the propagation of chaos

The result proved in this subsection is used in Section 3 in order to provide estimates independent of ϵ for the H^k-norm of the expressions (5) and (6). We use the standard Sobolev space notation $H^k := H^k(\mathbb{T})$, for $k \in \mathbb{N}$, and also $L^p := L^p(\mathbb{T})$, for $p \in [1, \infty]$. As already mentioned, we will always assume a scaling of type $N\epsilon^\theta = 1$, for θ large enough, say $\theta > \theta_0$. In this paper, we are not interested in optimising in θ (i.e., in finding its lowest admissible value).

Proposition 2.6. *Let the assumptions of Propositions 2.1 and 2.3 be satisfied, and let $\mathbb{N} \ni c \geq 2$. Then, in the regime $N\epsilon^\theta = 1$, for θ large enough, we have that*

$$\mathbb{E}\left[\left\|\frac{1}{N}\sum_{i=1}^{N} p_i^{n_1}(t)\frac{\partial^n}{\partial^n x}w_\epsilon(\cdot - q_i(t))\right\|_{L^c}^c\right] \tag{22a}$$

and

$$\mathbb{E}\left[\left\|\frac{1}{N}\sum_{i=1}^{N}\left\{\frac{1}{N}\sum_{j=1}^{N} W'(q_i(t) - q_j(t))\right\} p_i^{n_1}(t)\frac{\partial^n}{\partial^n x}w_\epsilon(\cdot - q_i(t))\right\|_{L^c}^c\right] \tag{22b}$$

are uniformly bounded in ϵ, N, and $t \in [0, T]$.

Even though the proof of Proposition 2.6 is a suitable extension of [4, Proof of Proposition 1.1], we include it here to keep the paper as self-contained as possible. For the benefit of the curious reader, we point out the analogies between the two proofs in the subsequent Remark 2.7, which may be skipped on a first reading.

Proof of Proposition 2.6. We first deal with (22a). Set $a_i(x,t) := p_i^{n_1}(t)\frac{\partial^n}{\partial^n x}w_\epsilon(x - q_i(t))$. If we expand the L^c-norm, we get

$$\begin{aligned}
&\mathbb{E}\left[\left\|\frac{1}{N}\sum_{i=1}^{N} p_i^{n_1}(t)\frac{\partial^n}{\partial^n x}w_\epsilon(\cdot - q_i(t))\right\|_{L^c}^c\right] \\
&\quad = \frac{1}{N^c}\sum_{\mathbf{j}\in\mathcal{S}_{1,c}}\mathbb{E}\left[\int_{\mathbb{T}}\prod_{k=1}^{c} a_{j_k}(x,t)\mathrm{d}x\right] + \frac{1}{N^c}\sum_{\mathbf{j}\in\mathcal{S}_{2,c}}\mathbb{E}\left[\int_{\mathbb{T}}\prod_{k=1}^{c} a_{j_k}(x,t)\mathrm{d}x\right],
\end{aligned}$$

where $\mathcal{S}_{1,c}$ and $\mathcal{S}_{2,c}$ are given by

$$\mathcal{S}_{1,c} := \{\mathbf{j} = (j_1, \ldots, j_c) \in \{1, \ldots, N\}^c : \mathbf{j} \text{ does not have repeated components}\}, \tag{23a}$$

$$\mathcal{S}_{2,c} := \{\mathbf{j} = (j_1, \ldots, j_c) \in \{1, \ldots, N\}^c : \mathbf{j} \text{ has repeated components}\}. \tag{23b}$$

We use the exchangeability of the particles, the fact that $\#\mathcal{S}_{2,c} \leq C(c)N^{c-1}$, the Hölder inequality, and the fact that all moments of p_i are uniformly bounded on $[0,T]$ (see Remark 2.5) to obtain

$$\frac{1}{N^c}\sum_{\mathbf{j}\in\mathcal{S}_{2,c}}\mathbb{E}\left[\int_{\mathbb{T}}\prod_{k=1}^{c} a_{j_k}(x,t)\mathrm{d}x\right] \leq \frac{C(c)}{N}\int_{\mathbb{T}}\mathbb{E}[a_1^c(x,t)]\mathrm{d}x \leq \frac{\mathbb{Q}(\epsilon^{-1})}{N} \to 0 \qquad \text{as } \epsilon \to 0, \tag{24}$$

where $\mathbb{Q}$ is some polynomial whose degree depends on n. The convergence to zero is granted by the scaling $N\epsilon^\theta = 1$, assuming that θ is large enough. For each $\mathbf{j} \in \mathcal{S}_{1,c}$, we now analyse $\mathbb{E}[\int_{\mathbb{T}}\prod_{k=1}^{c} a_{j_k}(x,t)\mathrm{d}x]$. The particles $\{(q_i, p_i)\}_{i=1}^{N}$ not being independent, we rely on the propagation of chaos, i.e., on Proposition 2.1. The strategy is the following: in each $a_{j_k}(x,t)$, we add and subtract relevant quantities associated with (3). More specifically, we split

$$p_i^{n_1}(t) = \underbrace{p_i^{n_1}(t) - \overline{p}_i^{n_1}(t)}_{A_{1,i}:=} + \underbrace{\overline{p}_i^{n_1}(t)}_{B_{1,i}:=}, \tag{25a}$$

$$\frac{\partial^n}{\partial^n x} w_\epsilon(x - q_i(t)) = \underbrace{\frac{\partial^n}{\partial^n x} w_\epsilon(x - q_i(t)) - \frac{\partial^n}{\partial^n x} w_\epsilon(x - \overline{q}_i(t))}_{A_{2,i}:=} + \underbrace{\frac{\partial^n}{\partial^n x} w_\epsilon(x - \overline{q}_i(t))}_{B_{2,i}:=}. \tag{25b}$$

The estimates

$$|A_{1,i}| \le C(n_1)|p_i(t) - \overline{p}_i(t)|(|p_i(t)|^{n_1-1} + |\overline{p}_i(t)|^{n_1-1}), \tag{26a}$$

$$|A_{2,i}| \le \mathbb{Q}(\epsilon^{-1})|q_i(t) - \overline{q}_i(t)|, \tag{26b}$$

$$|B_{2,i}| \le \mathbb{Q}(\epsilon^{-1}), \tag{26c}$$

where $\mathbb{Q}$ is a polynomial, follow easily from Taylor expansions and bounds on derivatives of w_ϵ. We regroup the 2^{2c} terms arising from the expansion of the product $\prod_{k=1}^{c}\left(A_{1,j_k} + B_{1,j_k}\right)\left(A_{2,j_k} + B_{2,j_k}\right)$ as

$$\prod_{k=1}^{c}\left(A_{1,j_k} + B_{1,j_k}\right)\left(A_{2,j_k} + B_{2,j_k}\right) = \prod_{k=1}^{c} B_{1,j_k}B_{2,j_k} + \sum_{s=1}^{2^{2c}-1} C_s,$$

where the sum spans all $2^{2c} - 1$ terms of the expansion which feature at least one factor of type A (i.e., each C_s is a product of $2c$ terms of type A and B, with at least one being of type A). As a result, we write

$$\mathbb{E}\left[\int_{\mathbb{T}} \prod_{k=1}^{c} a_{j_k}(x,t)\mathrm{d}x\right] = \mathbb{E}\left[\int_{\mathbb{T}} \prod_{k=1}^{c} B_{1,j_k}B_{2,j_k}\mathrm{d}x\right] + \sum_{s=1}^{2^{2c}-1} \mathbb{E}\left[\int_{\mathbb{T}} C_s \mathrm{d}x\right] := T_1 + T_2. \tag{27}$$

We bound T_2. As each term C_s contains a factor of type A, we can use (26) to deduce that

$$\begin{aligned}
|C_s| \le & \left(\prod_{i=1}^{c} |p_i(t) - \overline{p}_i(t)|^{\alpha_i} |q_i(t) - \overline{q}_i(t)|^{\beta_i}\right) \\
& \times \left(\prod_{i=1}^{c} \left[C(n_1)(|p_i(t)|^{n_1-1} + |\overline{p}_i(t)|^{n_1-1})\right]^{\alpha_i} \left[\mathbb{Q}(\epsilon^{-1})\right]^{\beta_i}\right) \\
& \times \left(\prod_{i=1}^{c} \left[\overline{p}_i(t)\right]^{1-\alpha_i} \left[\mathbb{Q}(\epsilon^{-1})\right]^{1-\beta_i}\right) =: T_3 \times T_4 \times T_5,
\end{aligned}$$

for some $\alpha_i, \beta_i \in \{0;1\}$, $\sum_{i=1}^{c} \alpha_i + \beta_i \in \{1, \ldots, 2c\}$. We can bound $\mathbb{E}[|C_s|]$ by applying a multi-factor Hölder inequality involving each term of the product $\mathbb{E}[T_3 \times T_4 \times T_5]$. More precisely, the expectation of each term of T_3 is either unitary, or dealt with by using Proposition 2.1 (propagation of chaos); the expectation of each term of T_4 and T_5 is either unitary, or dealt with by relying on the fact that all moments of $\overline{p}_i(t)$, $p_i(t)$ are uniformly bounded on $[0,T]$, see Remark 2.5. Due to the constraint $\sum_{i=1}^{c} \alpha_i + \beta_i \in \{1, \ldots, 2c\}$, we can apply Proposition 2.1 at least once. Thus $\mathbb{E}[|C_s|] \le C(n_1)N^{-\gamma_1}\epsilon^{-\gamma_2}$, for some $\gamma_1, \gamma_2 > 0$, for $s = 1, \ldots, 2^{2c} - 1$. Provided that θ is large enough, we deduce that $T_2 \to 0$ as $\epsilon \to 0$.

As for T_1, we rely on independence and identical distribution of the particles $\{(\overline{q}_i, \overline{p}_i)\}_{i=1}^{N}$ and write

$$\mathbb{E}[T_1] = \mathbb{E}\Big[\int_{\mathbb{T}} \prod_{k=1}^{c} \overline{p}_i^{n_1}(t)\frac{\partial^n}{\partial^n x} w_\epsilon(x-\overline{q}_i(t))\mathrm{d}x\Big]$$
$$\leq \int_{\mathbb{T}} \Big|\mathbb{E}\Big[\overline{p}_1^{n_1}(t)\frac{\partial^n}{\partial x^n} w(x-\overline{q}_1(t))\Big]\Big|^c \mathrm{d}x \leq C(t_0, f_0, n),$$

where the last inequality is given by Proposition 2.3. The expectation in (22a) is thus dealt with.

As for the expectation in (22b), the analysis proceeds similarly, and we only sketch the relevant details. We may think of the argument of the L^c-norm as a sum over two indexes $i, j = 1, \dots, N$, thus defining $a_{i,j}(x,t) := W'(q_i(t)-q_j(t))p_i^{n_1}(t)\frac{\partial^n}{\partial^n x}w_\epsilon(x-q_i(t))$. We split the L^c-norm expansion into the contributions given over the index sets $\mathcal{S}_{1,2c}$ and $\mathcal{S}_{2,2c}$ (c couples of indexes). The expectation associated with the index set $\mathcal{S}_{2,2c}$ vanishes in the limit $\epsilon \to 0$, using the same arguments leading to (24). Now fix $\mathbf{j} \in S_{1,2c}$. If we add the rewriting

$$W'(q_i(t)-q_j(t)) = \underbrace{W'(q_i(t)-q_j(t)) - W'(\overline{q}_i(t)-\overline{q}_j(t))}_{A_{3,i,j}:=} + \underbrace{W'(\overline{q}_i(t)-\overline{q}_j(t))}_{B_{3,i,j}:=}$$

to those in (25), with the associated bound

$$|A_{3,i,j}| \leq C(W)\left\{|q_i(t)-\overline{q}_i(t)| + |q_j(t)-\overline{q}_j(t)|\right\}$$

we may then write

$$\mathbb{E}\Big[\int_{\mathbb{T}} \prod_{k=1}^{c} a_{j_{2k-1},j_{2k}}(x,t)\mathrm{d}x\Big] = \mathbb{E}\Big[\int_{\mathbb{T}} \prod_{k=1}^{c} B_{1,j_{2k-1}}B_{2,j_{2k-1}}B_{3,j_{2k-1},j_{2k}}\mathrm{d}x\Big] + \sum_{s=1}^{2^{3c}-1}\mathbb{E}\Big[\int_{\mathbb{T}} C_s \mathrm{d}x\Big]$$
$$=: T_1 + T_2, \quad (28)$$

where the notation is in analogy to (27). The convergence $T_2 \to 0$ is settled as in the first part of the proof, and we omit the details. To bound T_1, we simply need to bound

$$\int_{\mathbb{T}} \Big|\mathbb{E}\Big[W'(\overline{q}_1(t)-\overline{q}_2(t))\overline{p}_1^{n_1}(t)\frac{\partial^n}{\partial^n x}w_\epsilon(x-\overline{q}_1(t))\Big]\Big|^c \mathrm{d}x, \quad (29)$$

where we have used again independence and identical distribution of the particles $\{(\overline{q}_i, \overline{p}_i)\}_{i=1}^N$. We notice that

$$\mathbb{E}\Big[W'(\overline{q}_1(t)-\overline{q}_2(t))\overline{p}_1^{n_1}(t)\frac{\partial^n}{\partial^n x}w_\epsilon(x-\overline{q}_1(t))\,|\overline{q}_1(t)=z_q, \overline{p}_1(t)=z_p\Big]$$
$$= z_p^{n_1}\frac{\partial^n}{\partial^n x}w_\epsilon(x-z_q)\mathbb{E}[W'(z_q-\overline{q}_2(t))\,|\overline{q}_1(t)=z_q, \overline{p}_1(t)=z_p]$$
$$= z_p^{n_1}\frac{\partial^n}{\partial^n x}w_\epsilon(x-z_q)\mathbb{E}[W'(z_q-\overline{q}_2(t))] = z_p^{n_1}\frac{\partial^n}{\partial^n x}w_\epsilon(x-z_q)W' * \mu_t(z_q),$$

which implies

$$\mathbb{E}\Big[W'(\overline{q}_1(t)-\overline{q}_2(t))\overline{p}_1^{n_1}(t)\frac{\partial^n}{\partial^n x}w_\epsilon(x-\overline{q}_1(t))\Big] = \mathbb{E}\Big[W' * \mu_t(\overline{q}_1(t))\overline{p}_1^{n_1}(t)\frac{\partial^n}{\partial^n x}w_\epsilon(x-\overline{q}_1(t))\Big].$$

The above equality shows that (29) is of the form prescribed by Proposition 2.3, for $g := W' * \mu_t$; as a matter of fact, $W' * \mu_t \in C^n$ because of the uniform regularity of μ_t for $t \in [0, T]$, see [28, (17.2)]. This ends the proof. $\square$

Remark 2.7. The proof of Proposition 2.6 is built on two splittings. The first one separates the index set in $\mathcal{S}_{1,c}$, $\mathcal{S}_{2,c}$ (and also $\mathcal{S}_{1,2c}$, $\mathcal{S}_{2,2c}$); the second one distinguishes terms of type A and B for every element in $\mathcal{S}_{1,c}$ (and also in $\mathcal{S}_{1,2c}$). The first splitting benefits from scaling arguments (in N, ϵ) which are found also in [4, Proposition 1.1] (see the distinction between terms ct, and I_1–I_4 therein). The second splitting benefits from Propagation of chaos, and does not have a counterpart in [4, Proposition 1.1].

Remark 2.8. In the proof of Proposition 2.6, the minimum power α that we need to employ when using the propagation of chaos is $\alpha = 2c$ (for (22a)) and power $\alpha = 3c$ (for (22b)). In the case of (22a), this can be seen easily from the multi-factor Hölder inequality used to deal with the one term $\mathbb{E}[|C_s|]$ for which $\sum_{i=1}^{c} \alpha_i + \beta_i = 2c$. An analogous consideration holds for (22b). This justifies the need for the propagation of chaos for $\alpha > 2$.

3 Evolution of the weakly interacting particle system

We analyse the time evolution of the densities (5)–(6) and start by deriving the relevant evolution equations.

Lemma 3.1. *The evolution equations for $\rho_\epsilon, j_\epsilon$, and $j_{2,\epsilon}$ are given by*

$$\frac{\partial \rho_\epsilon}{\partial t}(x,t) = -\frac{\partial j_\epsilon}{\partial x}(x,t), \tag{30a}$$

$$\begin{aligned}\frac{\partial j_\epsilon}{\partial t}(x,t) = &-\gamma j_\epsilon(x,t) - j_{2,\epsilon}(x,t) - \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{N}\sum_{j=1}^{N} W'(q_i(t)-q_j(t))\right) w_\epsilon(x-q_i(t)) \\ &+ \underbrace{\frac{\sigma}{N}\sum_{i=1}^{N} w_\epsilon(x-q_i(t))\dot\beta_i}_{=:\dot{\mathcal{Z}}_N(x,t)},\end{aligned} \tag{30b}$$

$$\begin{aligned}\frac{\partial j_{2,\epsilon}}{\partial t}(x,t) = &-2\gamma j_{2,\epsilon}(x,t) - j_{3,\epsilon}(x,t) - \frac{2}{N}\sum_{i=1}^{N}\left(\frac{1}{N}\sum_{j=1}^{N} W'(q_i(t)-q_j(t))\right) p_i(t) w'_\epsilon(x-q_i(t)) \\ &+ \sigma^2 \frac{\partial \rho_\epsilon}{\partial x}(x,t) + \frac{\sigma}{N}\sum_{i=1}^{N} 2p_i(t) w'_\epsilon(x-q_i(t))\dot\beta_i,\end{aligned} \tag{30c}$$

where $j_{3,\epsilon} := N^{-1}\sum_{i=1}^{N} p_i^3(t) w''_\epsilon(x-q_i(t))$.

The proof of the lemma above is a simple application of the Itô formula, and thus omitted.

3.1 Compactness argument

We now turn to the main result of this section.

Proposition 3.2. *Let $T > 0$. Let the assumptions of Propositions 2.1 and 2.3 be satisfied. Assume the scaling $N\epsilon^\theta = 1$, for θ large enough. The families of processes $\{\rho_\epsilon\}_\epsilon$, $\{j_\epsilon\}_\epsilon$, and $\{j_{2,\epsilon}\}_\epsilon$ are tight (hence relatively compact in distribution) in $C(0,T;L^2)$, as $\epsilon \to 0$.*

Proof. Assume for the time being (we will show this below) that

$$\mathbb{E}[\|\rho_\epsilon\|_{\mathscr{U}}], \quad \mathbb{E}[\|j_\epsilon\|_{\mathscr{U}}], \quad \mathbb{E}[\|j_{2,\epsilon}\|_{\mathscr{U}}] \quad \text{are uniformly bounded as } \epsilon \to 0, \tag{31}$$

where $\|\cdot\|_{\mathscr{U}}$ is the natural norm of the space

$$\mathscr{U} := L^\infty(0,T;H^1) \cap C^\beta(0,T;H^{-1}), \quad \text{for some } \beta \in (0,1/2). \tag{32}$$

Using [26, Theorem 5], it is straightforward to deduce that the embedding $\mathscr{U} \hookrightarrow \mathscr{Z} := C(0,T;L^2)$ is compact. In addition, the sets $G_j := \{u \in \mathscr{U} : \|u\|_{\mathscr{U}} \leq j\}$ are compact in $\mathscr{Z}$, for each $j \in \mathbb{N}$. Now fix $a > 0$. If we denote the law of ρ_ϵ by χ_ϵ, we get

$$\chi_\epsilon(\mathscr{Z} \setminus G_j) = \int_{\mathscr{Z}\setminus G_j} \chi_\epsilon(\mathrm{d}\rho) = \int_{\mathscr{U}\setminus G_j} \chi_\epsilon(\mathrm{d}\rho) \leq \frac{1}{j}\int_{\mathscr{U}} \|\rho\|_{\mathscr{U}}\chi_\epsilon(\mathrm{d}\rho) \leq a$$

for all $\epsilon \in (0,1]$, provided that j is large enough, thanks to (31). An analogous argument applies to $\{j_\epsilon\}_\epsilon$ and $\{j_{2,\epsilon}\}_\epsilon$. This corresponds to tightness for the families $\{\rho_\epsilon\}_\epsilon$, $\{j_\epsilon\}_\epsilon$, and $\{j_{2,\epsilon}\}_\epsilon$, hence the Prokhorov Theorem [14, Theorem 14.3] is applicable and gives relative compactness in distribution for the three families. In order to complete the proof, we need to show (31).

Uniform bounds for $\{\rho_\epsilon\}_\epsilon$. We show that

$$\mathbb{E}\Big[\|\rho_\epsilon\|_{L^\infty(0,T;H^1)}\Big] \leq C, \tag{33a}$$

$$\mathbb{E}\Big[\|\rho_\epsilon\|_{C^\beta(0,T;H^{-1})}\Big] \leq C, \tag{33b}$$

for a constant C, independent of ϵ and N. Using (30a), we deduce

$$\|\rho_\epsilon\|^2_{L^\infty(0,T;H^1)} = \sup_{t\in[0,T]} \|\rho_\epsilon(\cdot,t)\|^2_{H^1} \leq 2\|\rho_\epsilon(\cdot,0)\|^2_{H^1} + 2T\int_0^T \|j_\epsilon(\cdot,s)\|^2_{H^2}\mathrm{d}s.$$

Estimate (33a) is then settled by invoking Proposition 2.6. We now take $v \in H^1$ and compute

$$\begin{aligned}
|\langle \rho_\epsilon(\cdot,t) - \rho_\epsilon(\cdot,s), v\rangle_{L^2}| &= \left|\int_{\mathbb{T}} [\rho_\epsilon(x,t) - \rho_\epsilon(x,s)]\, v(x)\mathrm{d}x\right| = \left|\int_{\mathbb{T}} \left(\int_s^t -\nabla\cdot j_\epsilon(x,z)\mathrm{d}z\right) v(x)\mathrm{d}x\right| \\
&= \left|\int_{\mathbb{T}} \left(\int_s^t j_\epsilon(x,z)\mathrm{d}z\right) \nabla v(x)\mathrm{d}x\right| \leq \left\|\int_s^t j_\epsilon(\cdot,z)\mathrm{d}z\right\|_{L^2} \|v\|_{H^1} \\
&\leq |t-s|^{\frac{1}{2}} \left(\int_s^t \|j_\epsilon(\cdot,z)\|^2_{L^2}\,\mathrm{d}z\right)^{\frac{1}{2}} \|v\|_{H^1}.
\end{aligned} \tag{34}$$

The bound $x \leq 1 + x^2$ valid for any $x \in \mathbb{R}$, the definition of the usual norm of $C^\beta(0,T;H^{-1})$, and (34) imply

$$\mathbb{E}\Big[\|\rho_\epsilon\|_{C^\beta(0,T;H^{-1})}\Big] \leq C + C\,\mathbb{E}\left[\|\rho_\epsilon(\cdot,0)\|^2_{L^2} + \int_0^T \|j_\epsilon(\cdot,z)\|^2_{L^2}\,\mathrm{d}z\right] \leq C, \tag{35}$$

for some $\beta \in (0,1/2)$, where the last inequality follows from Proposition 2.6. We have thus proved (33b).

Uniform bounds for $\{j_\epsilon\}_\epsilon$. Again, we show that there exists a constant C, independent of ϵ and

N, such that

$$\mathbb{E}\Big[\|j_\epsilon\|_{L^\infty(0,T;H^1)}\Big] \le C, \tag{36a}$$

$$\mathbb{E}\Big[\|j_\epsilon\|_{C^\beta(0,T;H^{-1})}\Big] \le C. \tag{36b}$$

We use (30b) and deduce that

$$\begin{aligned}
\|j_\epsilon\|^2_{L^\infty(0,T;H^1)} &= \sup_{t\in[0,T]} \|j_\epsilon(\cdot,t)\|^2_{H^1} \\
&\le C\Bigg\{\|j_\epsilon(\cdot,0)\|^2_{H^1} + \gamma\int_0^T \|j_\epsilon(\cdot,z)\|^2_{H^1}\mathrm{d}z + \int_0^T \|j_{2,\epsilon}(\cdot,z)\|^2_{H^1}\mathrm{d}z \\
&\quad + \int_0^T \left\| \frac{1}{N}\sum_{i=1}^N \left(\frac{1}{N}\sum_{j=1}^N W'(q_i(z)-q_j(z))\right) w_\epsilon(\cdot - q_i(z))\right\|^2_{H^1} \mathrm{d}z \\
&\quad + \sup_{t\in[0,T]} \left\| \int_0^t \frac{\sigma}{N}\sum_{i=1}^N w_\epsilon(\cdot - q_i(z))\mathrm{d}\beta_i\right\|^2_{H^1}\Bigg\} =: T_1 + \cdots + T_5.
\end{aligned}$$

Uniform bounds for $\mathbb{E}[T_1], \mathbb{E}[T_2], \mathbb{E}[T_3]$, and $\mathbb{E}[T_4]$ are directly given by Proposition 2.6. As for $\mathbb{E}[T_5]$, we invoke [5, Theorem 4.36] and bound

$$\begin{aligned}
\mathbb{E}[T_5] &\le C\,\mathbb{E}\left[\int_0^T \sum_{i=1}^N \left\|\frac{\sigma}{N} w_\epsilon(\cdot - q_i(s))\right\|^2_{H^1}\mathrm{d}s\right] = C\int_0^T \frac{\sigma^2}{N^2}\sum_{i=1}^N \mathbb{E}\Big[\|w_\epsilon(\cdot - q_i(s))\|^2_{H^1}\Big]\mathrm{d}s \\
&\le CT\frac{\sigma^2}{N^2}N\left(\frac{1}{\epsilon}+\frac{1}{\epsilon^3}\right) \le \frac{CT\sigma^2}{N\epsilon^3},
\end{aligned}$$

where the reader is also referred to [4, Proof of Proposition 1.1] for the scalings of Sobolev norms of $w_\epsilon(\cdot - q_i(s))$, which we have used in the second line above. Estimate (36a) is thus established. In order to prove (36b), we analyse the quantity $|\langle j_\epsilon(\cdot,t) - j_\epsilon(\cdot,s), v\rangle_{H^{-1},H^1}|$. Bounding the relevant contributions coming from the initial datum and the three deterministic integrands is analogous to (34)–(35). As for the stochastic noise, we rely on [11, Lemma 2.1] and write, for $\alpha \in (0,1/2)$ and $\lambda > 2$ satisfying $\alpha\lambda > 1$,

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_0^\cdot \frac{\sigma}{N}\sum_{i=1}^N w_\epsilon(\cdot - q_i(t))\mathrm{d}\beta_i(s)\right\|^\lambda_{W^{\alpha,\lambda}(0,T;H^{-1})}\right] \\
&\quad \le C(\alpha,\lambda)\mathbb{E}\left[\int_0^T \frac{\sigma^\lambda}{N^\lambda}\left(\sum_{i=1}^N \|w_\epsilon(\cdot - q_i(s))\|^2_{L^2}\right)^{\lambda/2}\mathrm{d}s\right] \\
&\quad \le C(\alpha,\lambda)T\frac{\sigma^\lambda}{N^\lambda}\left(\frac{CN}{\epsilon}\right)^{\lambda/2} = \frac{C(\alpha,\lambda,\sigma)T}{(N\epsilon)^{\lambda/2}}.
\end{aligned}$$

We conclude the analysis for $\mathbb{E}[T_5]$ using the embedding $W^{\alpha,\lambda}(0,T;H^{-1}) \hookrightarrow C^\beta(0,T;H^{-1})$ for some $\beta \in (0, \alpha - 1/\lambda)$. This embedding is a consequence, e.g., of [7]. Thus (36b) is settled.

Uniform bounds for $\{j_{2,\epsilon}\}_\epsilon$. The argument is almost identical to that used for the family $\{j_\epsilon\}_\epsilon$.

We show that

$$\mathbb{E}\Big[\|j_{2,\epsilon}\|_{L^\infty(0,T;H^1)}\Big] \le C, \tag{37a}$$

$$\mathbb{E}\Big[\|j_{2,\epsilon}\|_{C^\beta(0,T;H^{-1})}\Big] \le C, \tag{37b}$$

for a constant C, independent of ϵ, N. We use (30c) and deduce that

$$\begin{aligned}
\|j_{2,\epsilon}\|^2_{L^\infty(0,T;H^1)} &= \sup_{t\in[0,T]} \|j_{2,\epsilon}(\cdot,t)\|^2_{H^1} \\
&\le C\bigg\{ \|j_{2,\epsilon}(\cdot,0)\|^2_{H^1} + \gamma\int_0^T \|j_{2,\epsilon}(\cdot,z)\|^2_{H^1}\mathrm{d}z + \int_0^T \|j_{3,\epsilon}(\cdot,z)\|^2_{H^1}\mathrm{d}z \\
&\quad + \int_0^T \left\| \frac{1}{N}\sum_{i=1}^N \left(\frac{1}{N}\sum_{j=1}^N W'(q_i(z)-q_j(z))\right) p_i(t) w'_\epsilon(\cdot - q_i(z)) \right\|^2_{H^1} \mathrm{d}z \\
&\quad + \sup_{t\in[0,T]} \left\| \int_0^t \frac{\sigma}{N}\sum_{i=1}^N p_i(t) w'_\epsilon(\cdot - q_i(z))\mathrm{d}\beta_i \right\|^2_{H^1} \bigg\} =: T_1 + \cdots + T_5.
\end{aligned}$$

The analysis involving the terms $T_1, \dots, T_4$ is analogous to that of the homonyms for $\{j_\epsilon\}_\epsilon$. We only need to deal with the stochastic noise. As for $\mathbb{E}[T_5]$,

$$\begin{aligned}
\mathbb{E}[T_5] &\le C\,\mathbb{E}\left[\int_0^T \sum_{i=1}^N \Big\|\frac{\sigma}{N} p_i(t) w'_\epsilon(\cdot - q_i(s))\Big\|^2_{H^1}\mathrm{d}s\right] = C\int_0^T \frac{\sigma^2}{N^2}\sum_{i=1}^N \mathbb{E}\Big[p_i^2(t)\,\|w'_\epsilon(\cdot - q_i(s))\|^2_{H^1}\Big] \\
&\le CT\frac{\sigma^2}{N^2} N\mathbb{E}[p_1^2(t)]\left(\frac{1}{\epsilon^3}+\frac{1}{\epsilon^5}\right) \le \frac{CT\sigma^2}{N\epsilon^5}.
\end{aligned} \tag{38}$$

For α and λ as in the previous part of the proof, we use the ℓ^p-Hölder inequality and bound

$$\begin{aligned}
&\mathbb{E}\left[\left\| \int_0^\cdot \frac{\sigma}{N}\sum_{i=1}^N p_i(t) w'_\epsilon(\cdot - q_i(t))\mathrm{d}\beta_i(s)\right\|^\lambda_{W^{\alpha,\lambda}(0,T;H^{-1})}\right] \\
&\le \frac{C(\alpha,\lambda)\sigma^\lambda}{N^\lambda}\int_0^T \mathbb{E}\left[\left(\sum_{i=1}^N \|p_i(s) w'_\epsilon(\cdot - q_i(s))\|^2_{L^2}\right)^{\lambda/2}\right]\mathrm{d}s \\
&\le \frac{C(\alpha,\lambda)\sigma^\lambda}{N^\lambda\epsilon^{3\lambda/2}}\int_0^T \mathbb{E}\left[\left(\sum_{i=1}^N p_i^2(s)\right)^{\lambda/2}\right]\mathrm{d}s \\
&\le \frac{C(\alpha,\lambda)\sigma^\lambda}{N^\lambda\epsilon^{3\lambda/2}}\int_0^T N^{\lambda/2-1}\mathbb{E}\left[\left(\sum_{i=1}^N p_i^\lambda(s)\right)\right]\mathrm{d}s = \frac{C(\alpha,\lambda)\sigma^\lambda}{N^{\lambda/2}\epsilon^{3\lambda/2}}\int_0^T \mathbb{E}\Big[p_1^\lambda(s)\Big]\mathrm{d}s = \frac{C(\alpha,\lambda,T)\sigma^\lambda}{N^{\lambda/2}\epsilon^{3\lambda/2}}.
\end{aligned} \tag{39}$$

Inequalities (38) and (39) allow us to deduce (37a) and (37b), and the proof is complete. □

Remark 3.3. In contrast to the methodology employed in [4, Proposition 1.1], which settles tightness in the case of independent particles, the proof of Proposition 3.2 does not rely on the Kolmogorov criterion. The reason is that the time regularity associated with the application of the propagation of chaos is not sufficiently high.

Remark 3.4. In principle, there is more than one natural choice for the definition of the space $\mathscr{U}$. Specifically, in (32), one might replace H^{-1} with any H^{-k}, where $k \in \mathbb{N}\cup\{0\}$, thus including L^2.

This would result in adapting estimate (34) in the case of $\{\rho_\epsilon\}_\epsilon$ (and analogous expressions in the case of $\{j_\epsilon\}_\epsilon$ and $\{j_{2,\epsilon}\}_\epsilon$), thus invoking Proposition 2.6 with a different parameter n. This directly reflects in a possibly different requirement for the scaling $N\epsilon^\theta = 1$. Since we are not concerned with the lowest admissible value of θ, the choice of H^{-1} is as good as any other of those listed above.

3.2 Approximating the interaction term

We show that the third term of the right-hand-side of (30b) is asymptotically equivalent (in the limit $\epsilon \to 0$ and $N \to 0$) to the nonlocal interaction term $\{W' * \rho_\epsilon\}\rho_\epsilon$.

Proposition 3.5. *Let $T > 0$. Let the assumptions of Propositions 2.1 and 2.3 be satisfied. Assume the scaling $N\epsilon^\theta = 1$, for θ large enough. We have the equality*

$$\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{N}\sum_{j=1}^{N} W'(q_i(t)-q_j(t))\right) w_\epsilon(x-q_i(t)) = \{W' * \rho_\epsilon(\cdot,t)\}(x)\rho_\epsilon(x,t) + r_{1,\epsilon}\rho_\epsilon(x,t) + r_{2,\epsilon}, \tag{40}$$

where r_1 and r_2 are stochastic remainders such that $|r_{1,\epsilon}| \le C(W)\sqrt{\epsilon}$ and $\mathbb{E}[|r_{2,\epsilon}|] \le C(W,f_0)\{\sqrt{\epsilon} + \epsilon^\beta\}$, for some $\beta = \beta(\theta) > 0$, and where f_0 is as in Proposition 2.3.

Before we prove the result above, we recall a simple lemma.

Lemma 3.6. *Let $f \in \mathcal{C}^0(\mathbb{T})$ be a Lipschitz function. There is a constant $C = C(f)$, independent of $\epsilon > 0$ and $a \in \mathbb{T}$, such that $|\int_\mathbb{T} w_\epsilon(y-a)f(y)\mathrm{d}y - f(a)| \le C\left(\sqrt{\epsilon} + \exp\{-C\epsilon^{-1}\}\right)$.*

Proof. Let $A_\epsilon := (a-\sqrt{\epsilon}, a+\sqrt{\epsilon})$. Since f is Lipschitz, we obtain

$$\begin{aligned}\int_\mathbb{T} w_\epsilon(y-a)f(y)\mathrm{d}y &= \int_{A_\epsilon} w_\epsilon(y-a)f(y)\mathrm{d}y + \int_{\mathbb{T}\backslash A_\epsilon} w_\epsilon(y-a)f(y)\mathrm{d}y\\ &\ge (f(a) - C\sqrt{\epsilon})\int_{A_\epsilon} w_\epsilon(y-a)\mathrm{d}y + \min_{x\in\mathbb{T}} f\int_{\mathbb{T}\backslash A_\epsilon} w_\epsilon(y-a)\mathrm{d}y\\ &\ge f(a)\left(1 - \int_{\mathbb{T}\backslash A_\epsilon} w_\epsilon(y-a)\mathrm{d}y\right) - C\sqrt{\epsilon} + \min_{x\in\mathbb{T}} f\int_{\mathbb{T}\backslash A_\epsilon} w_\epsilon(y-a)\mathrm{d}y.\end{aligned} \tag{41}$$

It is immediate to notice that $\int_{\mathbb{T}\backslash A_\epsilon} w_\epsilon(y-a)\mathrm{d}y \le C\exp\{-C\epsilon^{-1}\}$ for some $C > 0$. From (41), we obtain

$$\begin{aligned}\int_\mathbb{T} w_\epsilon(y-a)f(y)\mathrm{d}y - f(a) &\ge -f(a)\int_{\mathbb{T}\backslash A_\epsilon} w_\epsilon(y-a)\mathrm{d}y + \min_{x\in\mathbb{T}} f\int_{\mathbb{T}\backslash A_\epsilon} w_\epsilon(y-a)\mathrm{d}y - C\sqrt{\epsilon}\\ &\ge C\left(\min_{x\in\mathbb{T}} f - \max_{x\in\mathbb{T}} f\right)\exp\{-C\epsilon^{-1}\} - C\sqrt{\epsilon}.\end{aligned}$$

An analogous inequality (with opposite sign) may be obtained in a similar way, completing the proof. ☐

Proof of Proposition 3.5. We split the left-hand-side of (40) as $T_1 + T_2$, where

$$T_1 := \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{N}\sum_{j=1}^{N} W'(x-q_j(t))\right) w_\epsilon(x-q_i(t))$$

and

$$T_2 := \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{N}\sum_{j=1}^{N}\{W'(q_i(t)-q_j(t)) - W'(x-q_j(t))\}\right) w_\epsilon(x-q_i(t)).$$

As for T_1, we separate the sums in i and j and deduce

$$\begin{aligned}
T_1 &= \left(\frac{1}{N}\sum_{i=1}^{N} w_\epsilon(x-q_i(t))\right)\left(\frac{1}{N}\sum_{j=1}^{N} W'(x-q_j(t))\right) = \rho_\epsilon(x,t)\left(\frac{1}{N}\sum_{j=1}^{N} W'(x-q_j(t))\right) \\
&= \rho_\epsilon(x,t)\left(r_{1,\epsilon} + \frac{1}{N}\sum_{j=1}^{N}\int_{\mathbb{T}} W'(x-y)w_\epsilon(y-q_j(t))\mathrm{d}y\right) \\
&= \{W' * \rho_\epsilon(\cdot,t)\}\,\rho_\epsilon(x,t) + r_{1,\epsilon}\rho_\epsilon(x,t).
\end{aligned}$$

Lemma 3.6 gives $|r_{1,\epsilon}| \le C(W)\sqrt{\epsilon}$, where C is independent of x, t, ω. With the notation of (40), it holds that $r_{2,\epsilon} = T_2$. We use a Taylor expansion and bound

$$\begin{aligned}
|r_{2,\epsilon}| &\le \frac{1}{N^2}\sum_{i,j=1}^{N} |W'(q_i(t)-q_j(t)) - W'(x-q_j(t))|\, w_\epsilon(x-q_i(t)) \\
&\le \frac{C(W)}{N^2}\sum_{i,j=1}^{N} |x-q_i(t)|\, w_\epsilon(x-q_i(t)) = \frac{C(W)}{N}\sum_{i=1}^{N} |x-q_i(t)|\, w_\epsilon(x-q_i(t)) \\
&= \frac{C(W)}{N}\sum_{i=1}^{N} |x-\overline{q}_i(t)|\, w_\epsilon(x-\overline{q}_i(t)) \\
&\quad + \frac{C(W)}{N}\sum_{i=1}^{N} \{|x-q_i(t)|\, w_\epsilon(x-q_i(t)) - |x-\overline{q}_i(t)|\, w_\epsilon(x-\overline{q}_i(t))\} =: T_3 + T_4.
\end{aligned}$$

Since the particles are identically distributed, we have

$$\begin{aligned}
\mathbb{E}\big[|T_3|\big] &= C(W)\mathbb{E}\big[|x-q_1(t)|\, w_\epsilon(x-q_1(t))\big] \\
&= C(W)\int_{\mathbb{T}} |y-x|\, w_\epsilon(x-y) f_{\overline{q}}(t,y)\mathrm{d}y \le C(W,f_0)\sqrt{\epsilon},
\end{aligned}$$

where $f_{\overline{q}}(t,\cdot)$ is the probability density function of $\overline{q}(t)$, and f_0 is as in Proposition 2.3. The last inequality above is given by Lemma 3.6: in particular, the constant C does not depend on time, as $\sup_{t\ge 0, q\in\mathbb{T}} \frac{\partial}{\partial q} f_{\overline{q}}(t,q)$ is finite. To see this, one may apply [28, (17.2)] and [1, Theorem 4.12], with analogous considerations to those made in the proof of Proposition 2.3.

As for T_4, we use a Taylor expansion, the bounds $\max_{x\in\mathbb{T}} w_\epsilon(x) \le C\epsilon^{-1}$ and $\max_{x\in\mathbb{T}} |w'_\epsilon(x)| \le C\epsilon^{-2}$, and write

$$\begin{aligned}
\mathbb{E}\big[|T_4|\big] &= C(W)\mathbb{E}\big[|x-q_1(t)|\, w_\epsilon(x-q_1(t)) - |x-\overline{q}_1(t)|\, w_\epsilon(x-\overline{q}_1(t))\big] \\
&\le C(W)\mathbb{E}\big[|x-q_1(t)| \cdot |w_\epsilon(x-q_1(t)) - w_\epsilon(x-\overline{q}_1(t))|\big] \\
&\quad + C(W)\mathbb{E}\big[|q_1(t)-\overline{q}_1(t)| w_\epsilon(x-\overline{q}_1(t))\big] \\
&\le C(W)\epsilon^{-2}\mathbb{E}\big[|x-q_1(t)| \cdot |q_1(t)-\overline{q}_1(t)|\big] \\
&\quad + C(W)\epsilon^{-1}\mathbb{E}\big[|q_1(t)-\overline{q}_1(t)|\big] \le C(W)\epsilon^{\beta},
\end{aligned}$$

for some $\beta = \beta(\theta) > 0$, where the last inequality follows from the propagation of chaos (Proposition 2.1), and the scaling $N\epsilon^\theta = 1$. The bound for $r_{2,\epsilon}$ is established, and the proof is complete. $\square$

3.3 Noise comparison

We want to replace the stochastic noise of (30b) (previously referred to as $\mathcal{Z}_N$) with a noise closed in ρ_ϵ and j_ϵ. We suitably adapt [4, Subsections 3.2 and 3.3].

We first recall a useful fact. Let γ_ϵ be the probability density function of a Gaussian random variable with mean zero and variance ϵ^2. It is not difficult to show that, for $r_\epsilon := w_\epsilon - \gamma_\epsilon$, it holds that

$$\|r_\epsilon\|_{C^0(-\pi;\pi)} \le \epsilon^\alpha, \text{ for some } \alpha \in (0,1). \tag{43}$$

Proposition 3.7. *Let the assumptions of Propositions 2.1 and 2.3 be satisfied. Assume the scaling $N\epsilon^\theta = 1$, for θ large enough. We define the stochastic noise*

$$\dot{\mathcal{Y}}_N := \sigma N^{-1/2}\sqrt{\rho_{\epsilon/\sqrt{2}}}\,Q^{1/2}_{\sqrt{2}\epsilon}\xi,$$

where ξ is space-time white noise and $Q_{\sqrt{2}\epsilon}\colon L^2 \to L^2$ is the convolution operator with kernel $w_{\sqrt{2}\epsilon}$ (i.e., $\tilde{\xi}_\epsilon := Q^{1/2}_{\sqrt{2}\epsilon}\xi$ is an H^1-valued Q-Wiener process with covariance operator $Q_{\sqrt{2}\epsilon}$). For some positive $C = C(T)$, $c_1(\theta)$, and $c_2(\theta)$, and α as in (43)*, we have*

$$\begin{aligned}&\left|\mathbb{E}[\mathcal{Z}_N(x_1,t)\mathcal{Z}_N(x_2,t)] - \mathbb{E}[\mathcal{Y}_N(x_1,t)\mathcal{Y}_N(x_2,t)]\right|\\ &\quad\le \frac{C\sigma^2}{N}w_{\sqrt{2}\epsilon}(x_1-x_2)\times\left\{|x_1-x_2| + \epsilon^{c_1(\theta)} + \epsilon^\alpha + \epsilon^{c_2(\theta)}|x_1-x_2|^{1/2}\right\} + \frac{C\sigma^2}{N}\epsilon^\alpha.\end{aligned}$$

This result is an adaptation of [4, Proof of Theorem 1.3]. We sketch the proof below, and defer more technical considerations to Remark 3.8.

Proof of Proposition 3.7. In what follows, the residuals r_ϵ in (43) appear several times. We do not specify the argument, as ultimately only their C^0-norms will play a role. Set $m := (x_1+x_2)/2$. We use the multiplication rule for Gaussian kernels [4, Lemma A.4], the independence of the Brownian noises, and we apply (43) several times to obtain

$$\begin{aligned}&\mathbb{E}[\mathcal{Z}_N(x_1,t)\mathcal{Z}_N(x_2,t)]\\ &\quad= \mathbb{E}\left[\left(\int_0^t \frac{\sigma}{N}\sum_{i=1}^N w_\epsilon(x_1-q_i(u))\mathrm{d}\beta_i(u)\right)\left(\int_0^t \frac{\sigma}{N}\sum_{i=1}^N w_\epsilon(x_2-q_i(u))\mathrm{d}\beta_i(u)\right)\right]\\ &\quad= \frac{\sigma^2}{N^2}\mathbb{E}\left[\sum_{i=1}^N\int_0^t w_\epsilon(x_1-q_i(u))w_\epsilon(x_2-q_i(u))\mathrm{d}u\right]\\ &\quad= \frac{\sigma^2}{N^2}\mathbb{E}\left[\sum_{i=1}^N\int_0^t \left(\gamma_\epsilon(x_1-q_i(u))+r_\epsilon\right)\left(\gamma_\epsilon(x_2-q_i(u))+r_\epsilon\right)\mathrm{d}u\right]\\ &\quad= \frac{\sigma^2}{N^2}\mathbb{E}\left[\sum_{i=1}^N\int_0^t \gamma_{\sqrt{2}\epsilon}(x_1-x_2)\gamma_{\epsilon/\sqrt{2}}(m-q_i(u))\mathrm{d}u\right]\\ &\qquad+ \frac{\sigma^2}{N^2}\mathbb{E}\left[\sum_{i=1}^N\int_0^t \left\{r_\epsilon^2 + r_\epsilon\gamma_\epsilon(x_1-q_i(u)) + r_\epsilon\gamma_\epsilon(x_2-q_i(u))\right\}\mathrm{d}u\right].\end{aligned}$$

We use (43) to switch back to the von Mises kernels, and use the definition of $\rho_{\epsilon/\sqrt{2}}$ to obtain

$$
\begin{aligned}
&\big|\mathbb{E}[\mathcal{Z}_N(x_1,t)\mathcal{Z}_N(x_2,t)] - \mathbb{E}[\mathcal{Y}_N(x_1,t)\mathcal{Y}_N(x_2,t)]\big| \\
&\quad\le \Bigg|\frac{\sigma^2}{N}w_{\sqrt{2}\epsilon}(x_1-x_2)\int_0^t \mathbb{E}\Big[\rho_{\epsilon/\sqrt{2}}(m,u)\Big]\mathrm{d}u \\
&\qquad -\frac{\sigma^2}{N}w_{\sqrt{2}\epsilon}(x_1-x_2)\int_0^t \mathbb{E}\Big[\sqrt{\rho_{\epsilon/\sqrt{2}}(x_1,u)\rho_{\epsilon/\sqrt{2}}(x_2,u)}\Big]\mathrm{d}u\Bigg| \\
&\qquad +\Bigg|\frac{\sigma^2}{N^2}\mathbb{E}\Bigg[\sum_{i=1}^N\int_0^t\Big\{3r_\epsilon^2 + r_{\epsilon/\sqrt{2}}r_{\sqrt{2}\epsilon}\Big\}\,\mathrm{d}u\Bigg] + \frac{\sigma^2}{N^2}\mathbb{E}\Bigg[\sum_{i=1}^N\int_0^t\Big\{r_{\epsilon/\sqrt{2}}w_{\sqrt{2}\epsilon}(x_1-x_2)\Big\}\,\mathrm{d}u\Bigg] \\
&\qquad +\frac{\sigma^2}{N^2}\mathbb{E}\Bigg[\sum_{i=1}^N\int_0^t\Big\{r_\epsilon w_\epsilon(x_1-q_i(u)) + r_\epsilon w_\epsilon(x_2-q_i(u)) + r_{\sqrt{2}\epsilon}w_{\epsilon/\sqrt{2}}(m-q_i(u))\Big\}\,\mathrm{d}u\Bigg]\Bigg| \\
&\quad =: |A_1 - A_2| + |A_3+A_4+A_5|.
\end{aligned}
$$

The bound $|A_3+A_4+A_5| \le (C\sigma^2/N)\{\epsilon^\alpha + \epsilon^\alpha w_{\sqrt{2}\epsilon}(x_1-x_2)\}$ follows easily from (43). In order to control $|A_1 - A_2|$, it is sufficient to bound

$$
\mathbb{E}\Big[\Big|\rho_{\epsilon/\sqrt{2}}(m) - \sqrt{\rho_{\epsilon/\sqrt{2}}(x_1)\rho_{\epsilon/\sqrt{2}}(x_2)}\Big|\Big], \tag{44}
$$

where we have fixed $u \in [0,T]$, and dropped the time dependence for notational convenience. We bound the random variable in (44) as

$$
\Big|\rho_{\epsilon/\sqrt{2}}(m) - \sqrt{\rho^2_{\epsilon/\sqrt{2}}(m) + b(x_1,x_2)}\Big| \le \sqrt{|b(x_1,x_2)|}, \tag{45}
$$

where

$$
\begin{aligned}
b(x_1,x_2) := \rho_{\epsilon/\sqrt{2}}(m)\Big[\rho_{\epsilon/\sqrt{2}}(x_1) + \rho_{\epsilon/\sqrt{2}}(x_2) - 2\rho_{\epsilon/\sqrt{2}}(m)\Big] \\
+ (\rho_{\epsilon/\sqrt{2}}(x_1) - \rho_{\epsilon/\sqrt{2}}(m))(\rho_{\epsilon/\sqrt{2}}(x_2) - \rho_{\epsilon/\sqrt{2}}(m)).
\end{aligned}
$$

The Hölder inequality implies that $\mathbb{E}\big[\sqrt{|b(x_1,x_2)|}\big]$ is bounded by

$$
\begin{aligned}
&\mathbb{E}\Big[\rho^2_{\epsilon/\sqrt{2}}(m)\Big]^{1/4}\mathbb{E}\Big[\Big|\rho_{\epsilon/\sqrt{2}}(x_1) + \rho_{\epsilon/\sqrt{2}}(x_2) - 2\rho_{\epsilon/\sqrt{2}}(m)\Big|^2\Big]^{1/4} \\
&\quad + \mathbb{E}\Big[\Big|\rho_{\epsilon/\sqrt{2}}(x_1) - \rho_{\epsilon/\sqrt{2}}(m)\Big|^4\Big]^{1/8}\mathbb{E}\Big[\Big|\rho_{\epsilon/\sqrt{2}}(x_2) - \rho_{\epsilon/\sqrt{2}}(m)\Big|^4\Big]^{1/8} =: T_1T_2 + T_3T_4.
\end{aligned}
$$

We notice that

$$
\begin{aligned}
&\mathbb{E}\Bigg[\Bigg|\frac{1}{N}\sum_{i=1} w_\epsilon(x-\overline{q}_i(t))\Bigg|^c\Bigg] \\
&\quad = N^{-c}\sum_{\mathbf{j}\in\mathcal{S}_{1,c}}\mathbb{E}\Bigg[\prod_{k=1}^c w_\epsilon(x-\overline{q}_{j_k}(t))\Bigg] + N^{-c}\sum_{\mathbf{j}\in\mathcal{S}_{2,c}}\mathbb{E}\Bigg[\prod_{k=1}^c w_\epsilon(x-\overline{q}_{j_k}(t))\Bigg] \\
&\quad \le \mathbb{E}\big[w_\epsilon(x-\overline{q}_1(t))\big]^c + N^{-1}\epsilon^{-c} \le \|w_\epsilon(x-\cdot)\|^c_{L^1}\|f_{\overline{q}}(t,\cdot)\|^c_{L^\infty} + N^{-1}\epsilon^{-c} \\
&\quad = \|f_{\overline{q}}(t,\cdot)\|^c_{L^\infty} + N^{-1}\epsilon^{-c},
\end{aligned} \tag{46}
$$

where $f_{\overline{q}}(t,\cdot)$ is the probability density function of $\overline{q}(t)$, and f_0 is as in Proposition 2.3. As θ is large enough, and taking into account $\sup_{t\geq 0}\|f_{\overline{q}}(t,\cdot)\|^c_{L^\infty} < \infty$ (implied by assumptions of Proposition 2.3 thanks to [28, (17.2)]) we see that the left-hand side of (46) is uniformly bounded in ϵ, x, and t. We may now bound $T_1, \ldots, T_4$. We write

$$T_1 \leq K\mathbb{E}\Big[\overline{\rho}^2_{\epsilon/\sqrt{2}}(m)\Big]^{1/4} + K\mathbb{E}\Big[\Big|\rho_{\epsilon/\sqrt{2}}(m) - \overline{\rho}_{\epsilon/\sqrt{2}}(m)\Big|^2\Big]^{1/4},$$

where $\overline{\rho}_\epsilon$ is the smoothed density with respect to the particle system (3). The first term in the right-hand side above is bounded by (46), while the second is bounded using the propagation of chaos. As a result, $T_1 \leq C$.

As for T_2, again by adding and subtracting relevant evaluations of $\overline{\rho}_\epsilon$, we obtain

$$\begin{aligned} T_2 \leq\; & K\mathbb{E}\Big[\Big|\overline{\rho}_{\epsilon/\sqrt{2}}(x_1) + \overline{\rho}_{\epsilon/\sqrt{2}}(x_2) - 2\overline{\rho}_{\epsilon/\sqrt{2}}(m)\Big|^2\Big]^{1/4} + K\mathbb{E}\Big[\Big|\rho_{\epsilon/\sqrt{2}}(x_1) - \overline{\rho}_{\epsilon/\sqrt{2}}(x_1)\Big|^2\Big]^{1/4} \\ & + K\mathbb{E}\Big[\Big|\rho_{\epsilon/\sqrt{2}}(x_2) - \overline{\rho}_{\epsilon/\sqrt{2}}(x_2)\Big|^2\Big]^{1/4} + K\mathbb{E}\Big[\Big|\rho_{\epsilon/\sqrt{2}}(m) - \overline{\rho}_{\epsilon/\sqrt{2}}(m)\Big|^2\Big]^{1/4}. \end{aligned} \tag{47}$$

The first term in the right-hand side of (47) can bounded by $K|x_1 - x_2|$, using the same strategy used in [4, Adaptation of proof of Theorem 1.3]; the remaining ones are controlled using the propagation of chaos. As a result, we get $T_2 \leq K|x_1 - x_2| + \epsilon^{\gamma_1}$, for some $\gamma_1 = \gamma_1(\theta) > 0$. The analysis of T_3, T_4 is similar to that of T_2. In the case of T_3

$$\begin{aligned} T_3 \leq\; & K\mathbb{E}\Big[\Big|\overline{\rho}_{\epsilon/\sqrt{2}}(x_1) - \overline{\rho}_{\epsilon/\sqrt{2}}(m)\Big|^4\Big]^{1/8} + K\mathbb{E}\Big[\Big|\rho_{\epsilon/\sqrt{2}}(x_1) - \overline{\rho}_{\epsilon/\sqrt{2}}(x_1)\Big|^2\Big]^{1/4} \\ & + K\mathbb{E}\Big[\Big|\rho_{\epsilon/\sqrt{2}}(m) - \overline{\rho}_{\epsilon/\sqrt{2}}(m)\Big|^2\Big]^{1/4}. \end{aligned} \tag{48}$$

The first term in the right-hand side of (48) can bounded by $K\sqrt{|x_1 - x_2|}$, using the same strategy used in [4, Adaptation of proof of Theorem 1.3]; propagation of chaos controls the remaining ones. So $T_3 \leq K\sqrt{|x_1 - x_2|} + \epsilon^{\gamma_2}$, for some $\gamma_2 = \gamma_2(\theta) > 0$. The estimate for T_4 is the same, with the couple (x_1, m) replaced by (x_2, m).

Putting everything together, we obtain the bound

$$\mathbb{E}\Big[\Big|\rho_{\epsilon/\sqrt{2}}(m) - \sqrt{\rho^2_{\epsilon/\sqrt{2}}(m) - b(x_1, x_2)}\Big|\Big] \leq C\left\{|x_1 - x_2| + \epsilon^{c_1(\theta)} + \epsilon^{c_2(\theta)}|x_1 - x_2|^{1/2}\right\}, \tag{49}$$

where $c_1(\theta) := \min\{\gamma_1; 2\gamma_2\}$ and $c_2(\theta) := \gamma_2$. This concludes the proof. □

Remark 3.8. The error bound of Proposition 3.7 is less sharp than the one provided in [4, Theorem 1.3] in the following sense: firstly, the spatial term contributions in (49) are not quadratic. This is due to the use of the suboptimal bound (45), as clarified in [4, Remark 3.4]. More precisely, we do not have an analogue of [4, Proposition B.8] in the case of weakly interacting particles, so we can not use more precise bounds involving inverse powers of ρ_ϵ; secondly, the propagation of chaos produces stand-alone contributions in ϵ (vanishing as $\epsilon \to 0$); finally, the need to switch from von Mises to Gaussian kernels (and vice versa) produces additional contributions in ϵ (also vanishing as $\epsilon \to 0$).

4 The regularised model

While the equations (30a)–(30b) describe the 'exact' evolution of the relevant densities $(\rho_\epsilon, j_\epsilon)$ associated to the weakly interacting particle system (2), they are not, however, closable in $(\rho_\epsilon, j_\epsilon)$: more precisely, they contain three terms (specifically, $j_{2,\epsilon}$, $\mathcal{Z}_N$, and the nonlocal interaction term of (30b)) which can not be related directly to $(\rho_\epsilon, j_\epsilon)$. In this final section, under suitable assumptions, we derive and analyse an SPDE which approximates (30a)–(30b). We propose the following approximations associated with the three terms mentioned above, and we point out the extent to which they are valid.

Approximation 1. The interaction term in (30b) is replaced by $\{W' * \rho_\epsilon\}\rho_\epsilon$. Proposition 3.5 implies that this replacement gives a vanishing error (in the L^1 sense) as $\epsilon \to 0$.

Approximation 2. We replace $j_{2,\epsilon}$ with $\frac{\sigma^2}{2\gamma}\frac{\partial \rho_\epsilon}{\partial x}$. This has been done also in [4], and we adapt the essential details here. In local equilibrium, the probability density function of the couple $(q_i(t), p_i(t))$ is approximately separable in the two variables (as shown in [8, Corollary 3.2]). We can thus write $\mathbb{E}[j_{2,\epsilon}] = \mathbb{E}[p_1^2(t)]\mathbb{E}[\partial\rho_\epsilon/\partial x]$, which suggests the proposed replacement. In a small temperature regime (corresponding to $\sigma^2/(2\gamma) \ll 1$), we see that $\mathrm{Var}[p_i^2(t)] \leq C\sigma^4/(2\gamma)^2 \ll \sigma^2/(2\gamma) \approx \mathbb{E}[p_i^2(t)]$, see again [8, Corollary 3.2]. It is in this case sensible to replace p_i^2 with $\mathbb{E}[p_i^2]$, which means replacing $j_{2,\epsilon}$ with $\frac{\sigma^2}{2\gamma}\frac{\partial \rho_\epsilon}{\partial x}$.

Approximation 3. We replace $\mathcal{Z}_N$ with $\sigma N^{-1/2}\sqrt{\rho_\epsilon}\,\tilde{\xi}_\epsilon$. This is justified along the lines of [4], and we adapt the necessary details. First, we notice that $\mathcal{Z}_N$ and $\mathcal{Y}_N$ are asymptotically equivalent in distribution for $\epsilon \to 0$, as shown in Proposition 3.7. In addition, one can show that, for each $t \in [0,T]$, $\{\rho_\epsilon(\cdot,t)\}_\epsilon$ has a unique limit in L^2 as $\epsilon \to 0$. This can be seen be taking two sequences $\{a_n; N_n^a\}$, $\{b_n; N_n^b\}$ (both satisfying the usual θ-scaling) and using scaling arguments (similar to those used, for example, in (46)) and the propagation of chaos to show that $\mathbb{E}\left[\|\rho_{a_n}(\cdot,t) - \rho_{b_n}(\cdot,t)\|_{L^2}^2\right] \to 0$ as $a_n, b_n \to 0$. As a result, the two quantities $\rho_\epsilon(\cdot,t)$ and $\rho_{\epsilon/\sqrt{2}}(\cdot,t)$ coincide in the limit. Therefore, for $\epsilon \ll 1$, we consider $\sigma N^{-1/2}\sqrt{\rho_\epsilon}\,\tilde{\xi}_\epsilon$ in spite of $\mathcal{Y}_N$, thus obtaining the overall noise replacement.

These approximations give the following *regularised Dean–Kawasaki model* for interacting particles in undamped regime

$$
\begin{cases}
\dfrac{\partial \tilde{\rho}_\epsilon}{\partial t}(x,t) = -\dfrac{\partial \tilde{j}_\epsilon}{\partial x}(x,t), & \text{(50a)}\\[2mm]
\dfrac{\partial \tilde{j}_\epsilon}{\partial t}(x,t) = -\gamma \tilde{j}_\epsilon(x,t) - \left(\dfrac{\sigma^2}{2\gamma}\right)\dfrac{\partial \tilde{\rho}_\epsilon}{\partial x}(x,t) - \{W' * \tilde{\rho}_\epsilon(\cdot,t)\}\tilde{\rho}_\epsilon(\cdot,t) + \dfrac{\sigma}{\sqrt{N}}\sqrt{\tilde{\rho}_\epsilon(x,t)}\,\tilde{\xi}_\epsilon, & \text{(50b)}\\[2mm]
\tilde{\rho}_\epsilon(x,0) = \rho_0(x), \quad \tilde{j}_\epsilon(x,0) = j_0(x),
\end{cases}
$$

for $(x,t) \in \mathbb{T} \times [0,T]$, and where (ρ_0, j_0) is a suitable initial datum. We used the notation $(\tilde{\rho}_\epsilon, \tilde{j}_\epsilon)$ to distinguish the solution of the SPDE (50) from the smoothed (exact) densities $(\rho_\epsilon, j_\epsilon)$. We establish a high-probability existence and uniqueness result (in the sense of mild solutions) for (50). Following [4, Subsection 4.3], we smooth the coefficient function of the noise in (50b) and study the system

$$
\begin{cases}
\mathrm{d}X_\epsilon(t) = [AX_\epsilon(t) + \alpha(X_\epsilon(t))]\,\mathrm{d}t + B_{N,\delta}(X_\epsilon(t))\mathrm{d}W_\epsilon,\\
X_\epsilon(0) = X_0,
\end{cases}
\tag{51}
$$

for $X_\epsilon(t) := (\tilde{\rho}_\epsilon(\cdot,t), \tilde{j}_\epsilon(\cdot,t))$, $X_0 := (\rho_0, j_0)$, $\dot{W}_\epsilon := (0, \tilde{\xi}_\epsilon)$, and where A (respectively, α) is a linear (respectively, nonlinear) operator on $\mathcal{W} := H^1(\mathbb{T}) \times H^1(\mathbb{T})$ defined by

$$
AX_\epsilon(t) := \left(-\frac{\partial \tilde{j}_\epsilon}{\partial x}(\cdot,t),\ -\gamma\tilde{j}_\epsilon(\cdot,t) - \left(\frac{\sigma^2}{2\gamma}\right)\frac{\partial \tilde{\rho}_\epsilon}{\partial x}(\cdot,t)\right), \qquad \alpha(X_\epsilon(t)) := \left(0, -\{W' * \tilde{\rho}_\epsilon(\cdot,t)\}\tilde{\rho}_\epsilon(\cdot,t)\right),
$$

and $B_{N,\delta}\colon \mathcal{W} \to \{f\colon \mathcal{W} \to L^2 \times L^2\}$ is defined as $B_N((\rho, j))(a, b) := \sigma N^{-1/2}\,(0,\, h_\delta(|\rho|)\cdot b)$, for h_δ being a $C^2(\mathbb{R})$-regularisation of the square-root function on $[-\delta, \delta]$, for some $\delta > 0$. A mild solution to (51) on $[0, T]$ is a $\mathcal{W}$-valued predictable process $X_{\epsilon,\delta} = (\tilde{\rho}_{\epsilon,\delta}, \tilde{j}_{\epsilon,\delta})$ defined on $[0, T]$ such that $\mathbb{P}(\int_0^T \|X_{\epsilon,\delta}(s)\|^2_{\mathcal{W}}\mathrm{d}s) = 1$, and satisfying, for each $t \in [0, T]$

$$X_{\epsilon,\delta}(t) = S(t)X_0 + \int_0^t S(t-s)\alpha(X_{\epsilon,\delta}(s))\mathrm{d}s + \int_0^t S(t-s)B_{N,\delta}(X_{\epsilon,\delta}(s))\mathrm{d}W_\epsilon, \qquad \mathbb{P}\text{-a.s.}$$

where $\{S(t)\}_{t\geq 0}$ is the C_0-semigroup generated by A (see [4, Lemma 4.2]).

We first of all analyse the noise-free version of (51).

Lemma 4.1. *Fix $0 < c_1 < c_2$. Consider the system*

$$\begin{cases} \mathrm{d}X(t) = [AX(t) + \alpha(X(t))]\,\mathrm{d}t, \\ X(0) = X_0 := (\rho_0, j_0), \end{cases} \tag{52}$$

and assume that $\min_{x\in\mathbb{T}} \rho_0(x) > c_1$ and $\|X_0\|_{\mathcal{W}} < c_2$. Then (52) *has a unique local $\mathcal{W}$-valued mild solution $Z := (\rho_Z, j_Z)$ up to some $T > 0$, such that*

$$\min_{x\in\mathbb{T}} \rho_Z(x, s) > c_1 \text{ and } \|Z(s)\|_{\mathcal{W}} < c_2, \quad \text{for all } s \in [0, T]. \tag{53}$$

Proof. The operator A generates a C_0-semigroup of contractions on $\mathcal{W}$, see for example [4, Lemma 4.2]. In addition, α is locally Lipschitz and locally bounded. To see this, choose (u_1, v_1) and (u_2, v_2) in a $\mathcal{W}$-ball of radius n. Then, using the Sobolev embedding $H^1 \subset C^0$ and the boundedness of W' and W'', we obtain

$$\begin{aligned}
&\|\alpha((u_1, v_1)) - \alpha((u_2, v_2))\|^2_{\mathcal{W}} \\
&\quad = \|\{W' * u_1\}u_1 - \{W' * u_2\}u_2\|^2_{L^2} + \|(\partial/\partial x)\left(\{W' * u_1\}u_1 - \{W' * u_2\}u_2\right)\|^2_{L^2} \\
&\quad \leq C\Big\{\|\{W' * (u_1 - u_2)\}u_1\|^2_{L^2} + \|\{W' * u_2\}(u_1 - u_2)\|^2_{L^2} + \|\{W'' * (u_1 - u_2)\}u_1\|^2_{L^2} \\
&\qquad + \|\{W'' * u_2\}(u_1 - u_2)\|^2_{L^2} + \|\{W' * (u_1 - u_2)\}u_1'\|^2_{L^2} + \|\{W' * u_2\}(u_1' - u_2')\|^2_{L^2}\Big\} \\
&\quad \leq C(n, W)\,\|(u_1, v_1) - (u_2, v_2)\|^2_{\mathcal{W}}\,,
\end{aligned} \tag{54}$$

which is the local Lipschitz property for α. Local boundedness is settled with an analogous computation. We apply [27, Theorem 4.5] to deduce the existence of a unique local $\mathcal{W}$-valued mild solution $Z := (\rho_Z, j_Z)$ to (52) up to some $T > 0$. Since the solution is càdlàg by [27, Remark 4.6], using the Sobolev embedding $H^1 \subset C^0$, we can choose $T > 0$ so that (53) is satisfied. □

Lemma 4.2. *Let X_0 be a deterministic initial datum for* (51)*. Then* (51) *admits a unique local mild solution.*

Proof. This follows from [27, Theorem 4.5], since (i) A generates a C_0-semigroup of contractions on $\mathcal{W}$; (ii) α is locally Lipschitz and locally bounded, see Lemma 4.1; (iii) $B_{N,\delta}$ is locally Lipschitz and satisfies the linear growth condition, see [4, Lemma 4.5]; (iv) the noise W_ϵ is a $\mathcal{W}$-valued Q-Wiener process whose covariance operator $Q_{\sqrt{2}\epsilon}$ has rapidly decaying eigenvalues, see [4, Subsection 4.2]. □

Now let X_ϵ be the unique local mild solution to (51). For some positive constants T, δ, and k, we define two relevant stopping times associated with (51), namely

$$\tau_k := \inf\{t>0 : \|X_\epsilon(t)\|_{\mathcal{W}} \geq k\} \wedge T, \qquad \mu_\delta := \tau_k \wedge \inf\left\{t>0 : \min_{x\in\mathbb{T}} \tilde{\rho}_\epsilon(x,t) \leq \delta\right\}. \tag{55}$$

Lemma 4.3. *Fix $k>0$, $\delta>0$, and $T>0$. Let X_ϵ be the unique local mild solution to (51). The following statements hold:*

(a) The total mass of the system is conserved up to τ_k, i.e., $\int_{\mathbb{T}} \tilde{\rho}_\epsilon(x,s)\mathrm{d}x = \int_{\mathbb{T}} \rho_0(x,s)\mathrm{d}x$ for all $s \leq \tau_k$.

(b) There exists a constant $C = C(X_0, W)$ such that, for all $x\in\mathbb{T}$ and for all $s\leq \mu_\delta$

$$-C \leq W' * \tilde{\rho}_\epsilon(x,s) \leq C, \quad -C \leq W'' * \tilde{\rho}_\epsilon(x,s) \leq C. \tag{56}$$

Proof. (a) We consider the $\mathcal{W}$-inner product of the mild formulation of (51) with the constant element $\zeta := (1,0) \in \mathcal{D}(A^\star)$, the symbol * denoting the adjoint. As $A^\star\zeta = 0$, we trivially get that

$$\int_0^T \mathbb{E}\left[\int_0^t \|\langle S(t-s)B_{N,\delta}(X_\epsilon(s)), A^\star\zeta\rangle\|^2_{L_2(\mathcal{W},\mathbb{R})}\,\mathrm{d}s\right]\mathrm{d}t < \infty.$$

We define $\hat{\alpha} := \alpha \circ R_k$, where

$$R_k \colon \mathcal{W} \mapsto \mathcal{W} \colon y \mapsto \begin{cases} y, & \text{if } \|y\|_{\mathcal{W}} \leq k, \\ k\frac{y}{\|y\|_{\mathcal{W}}}, & \text{if } \|y\|_{\mathcal{W}} > k \end{cases}$$

is a standard retraction map. Since the map $\hat{\alpha}$ is Lipschitz continuous, we have a unique global mild solution $\hat{X}_\epsilon$ to (51) with α replaced by $\hat{\alpha}$, which then clearly satisfies $\mathbb{P}(\int_0^T \|\hat{X}_\epsilon(t)\|_{\mathcal{W}}\,\mathrm{d}t < \infty) = 1$. Since we have predictability of both the deterministic and stochastic integrands involved in the definition of mild solution (to (51) with α replaced by $\hat{\alpha}$), we follow the proof of [13, Proposition 2.10, part (ii)], but *only* with the specific choice of ζ made above (and *not* with any $\zeta \in \mathcal{D}(A^\star)$). We deduce that $\hat{X}_\epsilon$ satisfies, $\mathbb{P}$-a.s.

$$\langle \hat{X}_\epsilon, \zeta\rangle = \langle X_0,\zeta\rangle + \int_0^t \left[\langle \hat{X}_\epsilon(s), A^\star\zeta\rangle + \langle \hat{\alpha}(\hat{X}_\epsilon(s)),\zeta\rangle\right]\mathrm{d}s + \int_0^t \langle B_{N,\delta}(\hat{X}_\epsilon(s)),\zeta\rangle \mathrm{d}W_\epsilon(s) = \langle X_0,\zeta\rangle.$$

Uniqueness of mild solutions implies that $\hat{X}_\epsilon(s) = X_\epsilon(s)$ for all $s \leq \tau_k$, and the claim is settled. Notice that we have *not* proved that X_ϵ is a weak solution to (51).

(b) The potential W being smooth, there exists C such that $-C \leq W'(y-x) \leq C$ for all $x,y\in\mathbb{T}$. If $s \leq \mu_\delta$, then $\tilde{\rho}_\epsilon(y,s) > 0$ for every $y \in \mathbb{T}$. We deduce that $-C\tilde{\rho}_\epsilon(y,s) \leq W'(x-y)\tilde{\rho}_\epsilon(y,s) \leq C\tilde{\rho}_\epsilon(y,s)$, for all $y\in\mathbb{T}$. Since $\mu_\delta \leq \tau_k$, we can rely on (a) and integrate in y, thus deducing that $-C(X_0,W) \leq W' * \tilde{\rho}_\epsilon(x,s) \leq C(W,X_0)$ for all $x\in\mathbb{T}$ and for all $s \leq \mu_\delta$. An identical argument applies with W'' replacing W'. □

We now turn to the proof of our main existence and uniqueness result for (50). This result is an adapted version of [4, Proposition 4.10 and Theorem 1.4].

Theorem 4.4 (High-probability existence and uniqueness result)**.** *Fix $\nu \in (0,1)$, and fix $0<\delta< c_1 < c_2 < k$. Let $X_0 = (\rho_0, j_0) \in \mathcal{W}$ be a deterministic initial condition, such that $\min_{x\in\mathbb{T}} \rho_0(x) > c_1$ and $\|X_0\|_{\mathcal{W}} < c_2$, and let $T>0$ be as in the statement of Lemma 4.1. Assume the scaling $N\epsilon^\theta = 1$, for θ large enough. It is possible to choose a sufficiently large number of particles N such that there* *exists a unique $\mathcal{W}$-valued mild solution $X_\epsilon = (\tilde{\rho}_\epsilon, \tilde{j}_\epsilon)$ satisfying* (50)*, up to time T, on a set $F_\nu \in \mathcal{F}$ such that $\mathbb{P}(F_\nu) \geq 1-\nu$.*

Proof. Consider the time $t \wedge \mu_\delta$, for $t \in [0,T]$, with μ_δ defined in (55). Let X_ϵ and Z be the local mild solutions to (51) and (52), respectively. We subtract the mild solution expressions for $X_\epsilon(t\wedge\mu_\delta)$ and $Z(t \wedge \mu_\delta)$, thus obtaining

$$\begin{aligned} X_\epsilon(t\wedge\mu_\delta) - Z(t\wedge\mu_\delta) = &\int_0^{t\wedge\mu_\delta} S(t\wedge\mu_\delta - s)\left[\alpha(X_\epsilon(s)) - \alpha(Z(s))\right] \mathrm{d}s \\ &+ \int_0^{t\wedge\mu_\delta} S(t\wedge\mu_\delta - s) B_{N,\delta}(X_\epsilon(s)) \mathrm{d}W_\epsilon. \end{aligned} \tag{57}$$

We look for a small-noise regime estimate up to time $t \wedge \mu_\delta$. In order to do so, we first prove that

$$\|\alpha(X_\epsilon(s)) - \alpha(Z(s))\|^2_{\mathcal{W}} \leq K_1^2\left(W, \|\rho_0\|_{H^1}, T\right)\|X_\epsilon(s) - Z(s)\|^2_{\mathcal{W}}, \qquad \text{for all } s \leq \mu_\delta. \tag{58}$$

We reuse computation (54) and deduce

$$\begin{aligned} &\|\alpha(X_\epsilon(s)) - \alpha(Z(s))\|^2_{\mathcal{W}} \\ &\quad\leq 2\Big\{\|\{W' * (\rho_Z - \tilde{\rho}_\epsilon)\}\rho_Z\|^2_{L^2} + \|\{W' * \tilde{\rho}_\epsilon\}(\rho_Z - \tilde{\rho}_\epsilon)\|^2_{L^2} + \|\{W'' * (\rho_Z - \tilde{\rho}_\epsilon)\}\rho_Z\|^2_{L^2} \\ &\qquad + \|\{W'' * \tilde{\rho}_\epsilon\}(\rho_Z - \tilde{\rho}_\epsilon)\|^2_{L^2} + \|\{W' * (\rho_Z - \tilde{\rho}_\epsilon)\}\rho'_Z\|^2_{L^2} + \|\{W' * \tilde{\rho}_\epsilon\}(\rho'_Z - \tilde{\rho}'_\epsilon)\|^2_{L^2}\Big\} \\ &\quad=: T_1 + \cdots + T_6. \end{aligned} \tag{59}$$

For $s \leq \mu_\delta$, we bound the terms T_2, T_4, T_6 using Lemma 4.3, and we bound the terms T_1, T_3, T_5 using the Sobolev embedding $H^1 \subset C^0$ and Lemma 4.1. Estimate (58) is proved.

We are now in the position to provide the small-noise regime estimate for (57). We closely follow the proof of [4, Proposition 4.10]. Let $q > 2$. We use [5, Proposition 7.3] to deduce that, for some $K_2 = K_2\left(W, \|\rho_0\|_{H^1}, T, q\right)$ and some $K_3 = K_3(\sigma, \delta, T, q, k)$

$$\begin{aligned} &\mathbb{E}\left[\sup_{s\in[0,t]} \|X_\epsilon(s\wedge\mu_\delta) - Z(s\wedge\mu_\delta)\|^q_{\mathcal{W}}\right] \\ &\quad\leq K_2\,\mathbb{E}\left[\int_0^t \|X_\epsilon(u) - Z(u)\|^q_{\mathcal{W}}\,\mathbf{1}_{[0,\mu_\delta]}(u)\mathrm{d}u\right] \\ &\qquad + \mathbb{E}\left[\sup_{s\in[0,T]}\left\|\int_0^s S(s\wedge\mu_\delta - u)B_{N,\delta}(X_\epsilon(u))\mathbf{1}_{[0,\mu_\delta]}(u)\mathrm{d}W_\epsilon\right\|^q\right] \\ &\quad\leq K_2\int_0^t \mathbb{E}\left[\sup_{s\in[0,u]}\|X_\epsilon(s\wedge\mu_\delta) - Z(s\wedge\mu_\delta)\|^q_{\mathcal{W}}\right]\mathrm{d}u \\ &\qquad + K(\sigma,\delta,T,q)M^q(\epsilon,N)\mathbb{E}\left[\int_0^T (1 + \|X_\epsilon(u)\|^q_{\mathcal{W}})\mathbf{1}_{[0,\mu_\delta]}(u)\mathrm{d}u\right] \\ &\quad\leq K_2\int_0^t \mathbb{E}\left[\sup_{s\in[0,u]}\|X_\epsilon(s\wedge\mu_\delta) - Z(s\wedge\mu_\delta)\|^q_{\mathcal{W}}\right]\mathrm{d}u + K_3 M^q(\epsilon,N), \end{aligned} \tag{60}$$

where $M^q(\epsilon, N)$ was derived in [4, Lemma 4.5], and decays to 0 as $\epsilon \to 0$ for θ large enough. It is easy to deduce that

$$\mathbb{E}\left[\sup_{s\in[0,T]}\|X_\epsilon(s\wedge\mu_\delta)-Z(s\wedge\mu_\delta)\|_{\mathcal{W}}^q\right]\le K_3M^q(\epsilon,N)e^{TK_2}. \tag{61}$$

For some small enough $\eta>0$, define

$$S:=\left\{\omega\in\Omega:\sup_{s\in[0,T]}\|X_\epsilon(s\wedge\mu_\delta)-Z(s\wedge\mu_\delta)\|_{\mathcal{W}}^q\le\eta\right\}.$$

Using the Chebyschev inequality in (61), we deduce that there exists N large enough so that $\mathbb{P}(S)\ge 1-\nu$. If η is chosen small enough, for any $\omega\in S$, we have that $\mu_\delta=\tau_k=T$. If this was not the case, we would have one of the following contradictions: on one hand, if $\mu_\delta<\tau_k\le T$, since $\min_{x\in\mathbb{T}}\rho_Z(x,s)>c_1>\delta$ for all $s\in[0,T]$ thanks to Lemma 4.1, and since η is small enough, we can use the embedding $H^1\subset C^0$ to deduce that $\min_{x\in\mathbb{T}}\tilde\rho_\epsilon(x,\mu_\delta)>\delta$, contradicting the definition of μ_δ; on the other hand, if $\mu_\delta=\tau_k<T$, since $\|\rho_Z(s)\|_{\mathcal{W}}<c_2<k$ for all $s\in[0,T]$ thanks to Lemma 4.1, and since η is small enough, we can use the same embedding $H^1\subset C^0$ to deduce that $\|\tilde\rho_\epsilon(\tau_k)\|_{\mathcal{W}}<k$, contradicting the definition of τ_k. This concludes the proof. □

Remark 4.5. The main difference between this section and [4, Section 4] is the combination of a solution localisation via stopping times (needed because the interaction term $\{W'*\tilde\rho_\epsilon(\cdot,t)\}\tilde\rho_\epsilon(\cdot,t)$ is superlinear) and the conservation of mass, see Theorem 4.4 and Lemma 4.3.

Remark 4.6. The existence theory described in this subsection can be slightly simplified, as one could deduce the validity of (56) for all $x\in\mathbb{T}$ and all $s\le\tau_k$ (rather than for all $s\le\mu_\delta$). In this case, the bounding constants would depend on k (hence on $\|\rho_0\|_{H^1}$) rather than on $\int_{\mathbb{T}}\rho_0(x)\mathrm{d}x$, simply because of the embedding $H^1\subset C^0$. The proof of Theorem 4.4 could then be adapted by using the stopping time τ_k instead of μ_δ in the small-noise regime analysis leading up to (61), thus making the use of Lemma 4.3 superfluous.
However, Lemma 4.3 provides a lower constant K_2 for the benefit of (61). The reason for this can be deduced from (59). The bounds associated with $T_1,\dots,T_6$ are of the type

$$T_i\le C_i^2\|X_\epsilon(s)-Z(s)\|_{\mathcal{W}}^2,\qquad i\in\{1,\dots,6\},$$

where the constants C_i, $i\in\{1,\dots,6\}$, depend on $\|\rho_Z\|_{H^1}$ (or equivalently, on $\|\rho_0\|_{H^1}$ and T). However, the terms T_2,T_4, and T_6 can be controlled more precisely, as C_2, C_4, and C_6 can be computed with the initial mass $\int_{\mathbb{T}}\rho_0(x)\mathrm{d}x$ only (Lemma 4.3). In the case of an initial datum satisfying $\int_{\mathbb{T}}\rho_0(x)\mathrm{d}x\ll\|\rho_0\|_{H^1}$, this corresponds to obtaining a constant K_1^2 in (58) which is approximately half the one we would get if we did not rely on Lemma 4.3 to deal with T_2,T_4, and T_6; this is simply because $K_1^2=C_1^2+\cdots+C_6^2$, and $C_2^2+C_4^2+C_6^2$ would, in this case, be negligible compared to $C_1^2+C_3^2+C_5^2$. This is turn implies that the constant K_2 in (61) can be scaled down by a factor up to $2^{q/2}$. Overall, this gives a smaller right-hand-side in (61), which reflects into a lower number of particles needed to meet the requirements of Theorem 4.4.

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3.2. Conclusions

We discussed the derivation and analysis of a regularised Dean–Kawasaki model in the case of Langevin particles interacting via a pairwise potential W.

At least in its basic structure, the argument reflects the methodology presented in Chapter 2, where a system of independent particles was studied. However, in order to complement and generalise the contents of Chapter 2, a number of non-trivial additional features associated with the particles' interaction (and stochastic dependence) are needed, and we summarise them below.

The first feature is Proposition 2.1, a propagation of chaos result which quantifies the 'distance' between the interacting particles system $\{(q_i, p_i)\}_{i=1}^N$ in question, and an auxiliary McKean–Vlasov type system of independent particles $\{\overline{q}_i, \overline{p}_i\}_{i=1}^N$. More specifically, we obtain the moment estimates of the type

$$\mathbb{E}[|q_i(t) - \overline{q}_i(t)|^\alpha]^{\frac{1}{\alpha}} \leq C(\alpha, W, T)N^{-1/2}, \qquad \mathbb{E}[|p_i(t) - \overline{p}_i(t)|^\alpha]^{\frac{1}{\alpha}} \leq C(\alpha, W, T)N^{-1/2}, \tag{3.1}$$

for any $t \in [0, T]$ and any even integer α. Proposition 2.1, which is a suitable adaptation of [46, Theorem 1.2], is a key auxiliary result which is used in several points of the paper (namely, Propositions 2.5, 3.2, 3.5, and 3.7). In a nutshell, Proposition 2.1 allows to refer the analysis to the system of independent particles $\{\overline{q}_i, \overline{p}_i\}_{i=1}^N$ (thus making some tools from Chapter 2 applicable) by paying a decaying polynomial contribution in N, see (3.1).

A second feature is the tightness argument for the relevant densities $\{\rho_\epsilon\}_\epsilon$, $\{j_\epsilon\}_\epsilon$, and $\{j_{2,\epsilon}\}_\epsilon$, which is proved in Proposition 3.2. In contrast to the analogous result from Chapter 2 (Proposition 1.1. therein), we rely on Simon's rather than Kolmogorov's compactness criterion: this is due to a low time regularity entailed by the application of Proposition 2.1.

A third feature is the appearance of the distinctive superlinear term $\{W' * \rho_\epsilon\}\rho_\epsilon$ (see Lemma 3.5) which encodes the particles interaction.

Additional features include: an adaptation of the DK noise replacement (Proposition 3.7); the analysis for the resulting DK model (51), where mild solutions are constructed using techniques analogous to those proposed in Chapter 2, and where in addition we deploy localisation via stopping times and conservation of mass in the system in order to deal with the superlinear nature of the model (given by the interaction term $\{W' * \rho_\epsilon\}\rho_\epsilon$).

The conclusions we are able to draw are the same as those of Chapter 2, but in the case of much more general particle systems. The open questions for this chapter reflect those detailed in the conclusions of Chapter 2. In particular, as mentioned earlier, we address the issue of positivity of solutions for the DK class in the next chapter by focusing on non-conservative modifications of stochastic thin-film equations.

Chapter 4

A priori positivity of solutions to a non-conservative stochastic thin-film equation

We devote this chapter to the issue of positivity of solutions for members of the DK class by focusing on suitable non-conservative modifications of a stochastic thin-film equation. The preprint we present here is available on arXiv [10].

4.1. Outline of the Article

As the DK class is concerned with the description of particle systems through meaningful densities, it is imperative to investigate whether or not the general structure of the DK class preserves the positivity of such densities. Asked differently: for a given member of the class, if we start with a positive initial datum bounded away from zero, do we obtain an almost sure positive solution defined up to a specified, deterministic, final time T? To the best of our knowledge, the arguments for arguing for an affirmative answer to this question are, so far, substantially limited.

To begin with, we know from [34, 35] that the original DK equation (1.1) admits nothing more than atomic solutions. The positivity requirement of the solutions is encoded in them taking values in the space of positive measures. If one does not allow for a non-conservative correction to the equation, then no positive smooth solutions can be found.

As for regularisations of the Dean-Kawasaki model, positivity is settled in some cases. In [23], the authors rely on a kinetic solution formulation to prove the existence of positive solutions to a perturbed porous medium equation (with noise given in Stratonovich sense). In [47], the author proves existence of positive strong solutions to a stochastic conservation law featuring viscosity. In both these works, the author heavily rely on the regularity of the deterministic component (being a p-Laplacian and a Laplacian, respectively). In our works, presented in Chapters 2 and 3 (i.e., in [12, 11]) positivity is only achieved in a high probability sense, as we have a second-order in time model (namely, a perturbation of a wave equation).

As for thin-film models, we are only aware of existence of positive martingale solutions in the case of quadratic mobility [24, 27].

The work we present in this chapter provides further clarifications on the positivity issue of the DK class by focusing on the interplay between mobility coefficient (or equivalently, the noise) and relevant source potentials for non-conservative modifications of the thin-film equation. Our hope is that this work will prove useful in the future analysis of DK type equations which do not benefit from regularisations, but which do feature non-conservative components.

Section 2 is devoted to proving our main result, which is concerned with extending local-in-time positive solutions up to any finite time. This result builds on a technical lemma, whose proof is the bulk of the paper, and the content of Section 3. This technical lemma provides relevant uniform bounds for the solution. These bounds refer to the a localisation procedure, which determines 'how long' the solution takes to hit an arbitrary given positive barrier. A comparison with the results and methodology of [24] is given in Sections 4 and 5.

Appendix B: Statement of Authorship

<table>
<tr><th colspan="9">This declaration concerns the article entitled:</th></tr>
<tr><td colspan="9">A priori positivity of solutions to a non-conservative stochastic thin-film equation</td></tr>
<tr><td colspan="9">Publication status (tick one)</td></tr>
<tr><td>draft manuscript</td><td>X</td><td>Submitted</td><td></td><td>In review</td><td></td><td>Accepted</td><td></td><td>Published</td></tr>
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Author: Federico Cornalba</td></tr>
<tr><td>Candidate's contribution to the paper (detailed, and also given as a percentage).</td><td colspan="8">The author of the thesis is the main contributor to the computations (80%) and the presentation (80%) for this work.
The remainder accounts for fruitful discussions and suggestions provided by the Ph.D. supervisors of the author of the thesis.</td></tr>
<tr><td>Statement from Candidate</td><td colspan="8">This paper reports on original research I conducted during the period of my Higher Degree by Research candidature.</td></tr>
<tr><td>Signed</td><td colspan="4"></td><td>Date</td><td colspan="3">17.9.2019</td></tr>
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A priori positivity of solutions to a non-conservative stochastic thin-film equation

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Abstract

Stochastic conservation laws are often challenging when it comes to proving existence of non-negative solutions. In a recent work by J. Fischer and G. Grün (2018, *Existence of positive solutions to stochastic thin-film equations*, SIAM J. Math. Anal.), existence of positive martingale solutions to a conservative stochastic thin-film equation is established in the case of quadratic mobility. In this work, we focus on a larger class of mobilities (including the linear one) for the thin-film model. In order to do so, we need to introduce nonlinear source potentials, thus obtaining a non-conservative version of the thin-film equation. For this model, we assume the existence of a sufficiently regular local solution (i.e., defined up to a stopping time τ) and, by providing suitable conditions on the source potentials and the noise, we prove that such solution can be extended up to any $T > 0$ and that it is positive with probability one. A thorough comparison with the aforementioned reference work is provided.

Key words: thin-film equation, drift correction, Itô calculus, nonlinearity, a priori analysis.

AMS (MOS) Subject Classification: 60H15, 35R60, 35G20

1 Introduction

We are interested in stochastic equations driven by random noise in spatial divergence form. A wide class of these equations can be written as

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(m(u) \nabla \frac{\delta F[u]}{\delta u} \right) + \Gamma(u) + \nabla \cdot \left(\sigma \sqrt{m(u)} \mathcal{W} \right) =: \mathscr{D}_1 + \mathscr{D}_2 + \mathscr{S}, \tag{1}$$

in the non-negative unknown $u = u(x,t)$, for $x \in D \subset \mathbb{R}^d$ and $t > 0$. Equation (1) describes the evolution of a system made of a large number of particles. The particles are subject to a gradient-flow dynamics (governed by the free energy F featured in the first drift term $\mathscr{D}_1$), to a nonlinear source (given by $\Gamma(u) \equiv \mathscr{D}_2$), and to mesoscopic thermal fluctuations (stochastic term $\mathscr{S}$, comprising an infinite-dimensional noise $\mathcal{W}$ and a given scaling parameter $\sigma \neq 0$). The evolution of the system is described by the particle density u, which is naturally required to be non-negative. The drift component $\mathscr{D}_1$ and the noise term $\mathscr{S}$ satisfy a fluctuation-dissipation relation [2] which can be identified in the powers of the so-called *mobility coefficient* $m(u)$ being 1 in $\mathscr{D}_1$ and $\frac{1}{2}$ in $\mathscr{S}$, respectively.

When $m(u) \equiv u$ and $\Gamma \equiv 0$, equation (1) is known as the *Dean-Kawasaki* model [6, 10]. This model poses hard mathematical challenges, the first of which is proving existence of positive solutions up to some given time $T > 0$. The main difficulties in doing so reside in the nature of the stochastic noise $\mathscr{S}$. To start with, this noise lacks Lipschitz properties and spatial regularity. If, in addition, we assume $\mathcal{W}$ to be a space-time white noise (this is a relevant choice in the physics literature), then the only existence result we are aware of is the recent work [13]. More specifically, in the case of $F(u) := (N/2)\int_D u(x)\log(u(x))\mathrm{d}x$ (corresponding to the Gibbs-Boltzmann entropy functional with pre-factor $N/2 > 0$), a unique probability measure-valued solution exists if and only if $N \in \mathbb{N}$; however, in this case, the solution is trivial, and coincides with the empirical measure associated with N independent diffusion processes.

Again for $m(u) \equiv u$, and for a specific class of $\Gamma \neq 0$, existence of measure-valued martingale solutions to (1) is available in space dimension one, see the work of von Renesse and coworkers [15, 1, 11, 12]. These results are based on the application of Dirichlet form methods, as well as on the interaction between drift and noise in the context of the Wasserstein geometry over the space of square-integrable probability measures. We also mention [3] for a high-probability existence and uniqueness result for a regularised version of (1).

In this work we investigate a priori positivity of solutions, up to any chosen time $T > 0$, in the specific case of a non-conservative *thin-film* equation

$$\begin{cases} \mathrm{d}u = -\nabla\cdot(m(u)\nabla[\Delta u - W'(u)])\,\mathrm{d}t + (h(u)|\nabla u|^2 + g(u))\,\mathrm{d}t + \nabla\cdot\left(\sqrt{m(u)}\mathrm{d}\mathcal{W}\right), \\ u(x,0) = u_0(x) \end{cases} \tag{2}$$

set on the spatial domain $D := (0, 2\pi)$, on some finite time domain $[0, T]$, and on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. More precisely, we assume the existence of a sufficiently regular local solution to (2) (i.e., defined up to a random time $\tau \leq T$) and we show that it can be extended up to T while remaining positive with probability one. Above, $u_0\colon D \to [0, \infty)$ is a suitable positive initial datum, $\mathcal{W}$ is a noise white in time and coloured in space, m is the mobility coefficient, and W, h and g are given nonlinear source potentials. These potentials compensate the noise contribution whenever the solution comes close to the singular regimes (these being identified by vanishing or diverging density); this is thoroughly discussed in Sections 3 and 4. The precise nature of $\mathcal{W}$, W, h, m, and g is stated in Subsection 1.1 below. We highlight that (2) fits into the form prescribed by (1) with $F(u) := \int_D \{|\nabla u(x)|^2/2 + W(u(x))\}\,\mathrm{d}x$ and $\Gamma(u) := h(u)|\nabla u|^2 + g(u)$.

Existence of positive martingale solutions to (2) has been established in the conservative case ($g \equiv h \equiv 0$) in [7], for the case of quadratic mobility $m(u) = u^2$; this mobility results in a linear multiplicative stochastic noise. The case of general polynomial mobility, including the linear case $m(u) = u$ (corresponding to the noise $\mathscr{S}$ featured in the Dean-Kawasaki model), seems hard to study for the conservative thin-film equation, see [7] again. This is why we analyse (2) for a non-trivial drift component Γ. However, our drift component Γ is not justified, as in the case of [15, 1, 11, 12], by the aforementioned Wasserstein geometry setting. Instead, it is needed in order to deal with algebraic cancellations arising from the Itô calculus applied to relevant functionals of the solution, these functionals being primarily associated with positivity of the solution, which is our main interest here. We also stress the fact that we only pursue a purely analytical justification of our drift component Γ, and we consequently neglect any physical modelling at this stage.

The paper is organised as follows. Subsection 1.1 contains basic assumptions on the functional setting, on the stochastic noise $\mathcal{W}$, as well as a parametrisation of interest for the relevant nonlinear quantities m, W, h, and g. Section 2 contains the two main results of this paper, Proposition 2.1 and Theorem 2.2. More specifically, Theorem 2.2 (which is also proved in this section) is concerned with positivity of solutions to (2) up to time T, which is our main interest. Its proof builds upon

Proposition 2.1, a technical result whose lengthy proof is the topic of Section 3. Sections 4 compares the contents of this paper with the setting and conclusions of [7]. Section 5 illustrates the difficulties that one encounters when trying to prove existence of local solutions to (2) via an approximating Galerkin scheme in the case of general mobility m, and also explains why such a scheme is effective in the specific case of quadratic mobility [7]. We summarise our findings and conclusions in Section 6.

1.1 Setting and notation

We work in a periodic function setting on $D := (0, 2\pi)$. The noise $\mathcal{W}$ is white in time and coloured in space. Its covariance operator Q is diagonalisable on the eigenfunctions of the Laplace operator on D with periodic boundary conditions. These eigenfunctions are given by the trigonometric family

$$\{e_r\}_{r=0}^{\infty} := \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \frac{\sin(3x)}{\sqrt{\pi}}, \frac{\cos(3x)}{\sqrt{\pi}}, \cdots \right\}.$$

Using [14, Proposition 2.1.10], we write the noise as $\mathcal{W}(t,x,\omega) = \sum_{r=0}^{\infty} \sqrt{\lambda_r} e_r(x) \beta_r(t,\omega)$, where $\{\lambda_r\}_{r=0}^{\infty}$ are the eigenvalues of Q associated with $\{e_r\}_{r=0}^{\infty}$, and $\{\beta_r\}_{r=0}^{\infty}$ is a family of independent Brownian motions. We assume the eigenvalues of Q to be rapidly decaying, say $\lambda_r \leq a_1 e^{-a_2 r}$, where $a_1, a_2 > 0$, for all $r \in \mathbb{N}_0$.

For some $\epsilon \in (0,1)$, let $A_0 := (0, 1-\epsilon)$, $A_1 := [1-\epsilon, 1+\epsilon]$, $A_\infty := (1+\epsilon, \infty)$. The mobility m and the functions h and g are given by

$$m(u) := \begin{cases} u^{\gamma_1}, & \text{if } u \in A_0, \\ f_m(u), & \text{if } u \in A_1, \\ u^{\gamma_2}, & \text{if } u \in A_\infty, \end{cases} \quad h(u) := \begin{cases} B_h u^{-p_h}, & \text{if } u \in A_0, \\ f_h(u), & \text{if } u \in A_1, \\ -B_h u^{c_h}, & \text{if } u \in A_\infty, \end{cases}$$
$$g(u) := \begin{cases} B_g u^{-p_g}, & \text{if } u \in A_0, \\ f_g(u), & \text{if } u \in A_1, \\ -B_g u^{c_g}, & \text{if } u \in A_\infty, \end{cases} \tag{3}$$

while W is given by $W(u) = u^{-p}$. The functions m, h, g, and W are understood to be infinite when $u \leq 0$. In the above, $p, B_h, p_h, c_h, B_g, p_g, c_g, \gamma_1$, and γ_2 are positive constants, while the functions f_h, f_g, and f_m are such that W, h, g, and m belong to $\mathcal{C}^\infty(0,\infty)$. It is easy to choose f_h and f_m such that, for some $\delta > 0$

$$f_m(u) > \delta, \quad \text{for all } u \in A_1, \tag{4a}$$
$$f_h'(u) \leq -\delta B_h, \quad \text{for all } u \in A_1. \tag{4b}$$

The potentials W, h, and the mobility m are sketched in Figure 1, while the potential g is not sketched (as it is qualitatively identical to h). We defined h, g and m piecewise on A_0 and A_∞ in order to be able to treat low and large density regimes differently. The definitions on A_1 provide smoothness on $(0,\infty)$ for the quantities in (3). Our definitions of W, h, g, and m are justified as follows: the potential W pushes mass away from the repulsive singularity 0, while obeying the conservation of mass. The source potentials h and g introduce mass in the system whenever the density is too low, and remove mass whenever the density is too large. In the case of h, the rate at which the introduction/removal of mass occurs is proportional to $|\nabla u|^2$. The mobility accounts for different drift and noise magnitudes in the low and large density regimes.

We use the symbol L^p to denote the space $L^p(D)$. We use the symbol $W^{s,p}$ to denote the Sobolev space $W^{s,p}_{\text{per}}(D)$ of 2π-periodic functions on $\mathbb{R}$ having distributional derivates up to order s belonging

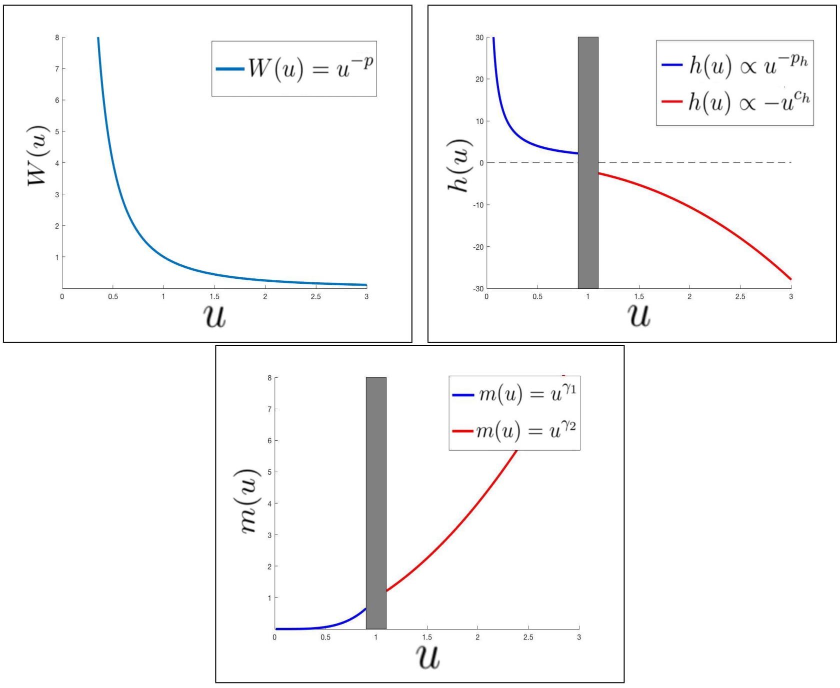

Figure 1: Sketches of W (top-left), h (top-right), and m (bottom). Plots on A_1 are not provided for h and m. The qualitative behaviour of g is identical to that of h.

to L^p. We abbreviate $H^s := W^{s,2}$. For a Hilbert space V, we use $\langle \cdot, \cdot \rangle_V$ and $\| \cdot \|_V$ to denote the V-inner product and V-norm, respectively. We drop the subscript if $V = L^2$. For a function u depending on space and time, we often write $u(t)$ instead of $u(x, t)$, and we indifferently use the notations u_x and ∇u to refer to spatial differentiation. Finally, C denotes a generic constant whose value may change from line to line; the dependency of this constant on specific parameters is highlighted whenever relevant.

2 A priori positivity of solutions

Let $T > 0$. We show that, if we assume the existence of a sufficiently regular solution to (2) up to a random time $\tau \leq T$, this solution can be extended up to T and is positive $\mathbb{P}$-a.s. In order to do so, we need the following auxiliary result.

Proposition 2.1. *Fix $T > 0$ and $\beta > 2$. Consider an initial datum $u_0 \in H^1$ such that $\delta_1 < \min_{x \in D} u_0(x)$ and $\|u_0\|_{H^1} < \delta_2$, for some $\delta_2 > \delta_1 > 0$, $\mathbb{P}$-a.s. We assume the existence of a $\mathbb{P}$-a.s. continuous H^1-valued strong solution u to* (2) *up to a random time $\tau \leq T$. In particular, the equation below is satisfied $\mathbb{P}$-a.s., for all $t > 0$*

$$u(t \wedge \tau) = u_0 + \int_0^{t \wedge \tau} \left[-\nabla \cdot (m(u) \nabla [\Delta u - W'(u)]) + (h(u)|\nabla u|^2 + g(u)) \right] \mathrm{d}s + \int_0^{t \wedge \tau} \nabla \cdot \left(\sqrt{m(u)}(\cdot) \right) \mathrm{d}\mathcal{W}. \tag{5}$$

Further, we assume that u is such that

$$\mathbb{P}\left(\int_0^\tau \|\nabla u(s)\|_{L^4}^4 \mathrm{d}s < \infty\right) = 1, \quad \mathbb{P}\left(\int_0^\tau \|\Delta u(s)\|^2 \mathrm{d}s < \infty\right) = 1,$$
$$\mathbb{E}\Big[\int_0^\tau \Big\|u_{xxx}(s)\sqrt{m(u(s))}\Big\|^2 \mathrm{d}s\Big] < \infty, \tag{6}$$

such that the deterministic integrand in (5) *is an* H^1*-valued predictable process, and such that the stochastic integrand in* (5) *is an* $L_2^0(H^1)$*-valued stochastically integrable process. Here* $L_2^0(H^1)$ *stands for the set of Hilbert-Schmidt operators from* $Q^{1/2}H^1$ *into* H^1*. For all* $n \in \mathbb{N}$ *such that* $n^{-1} < \delta_1$ *and* $n > \delta_2$*, we assume* $\tau_n \leq \tau \leq T$*, where the stopping time* τ_n *is given by*

$$\tau_n := \inf\left\{t > 0 : \min_{x \in D} u(x,t) \leq n^{-1}\right\} \wedge \inf\left\{t > 0 : \|u(t)\|_{H^1} \geq n\right\} \wedge T. \tag{7}$$

Assume the following conditions

$$\sum_{r=0}^{\infty} \lambda_r \text{ is small enough,} \tag{C1}$$
$$p_h, B_h, c_h \text{ are big enough,} \tag{C2}$$
$$p_g, B_g, c_g \text{ are big enough.} \tag{C3}$$

Let $F_1 \colon H^1 \to \mathbb{R} \cup \{\infty\} \colon u \mapsto \int_D |u|^{-\beta}$*, let* $F_2 \colon H^1 \to \mathbb{R} \colon u \mapsto \frac{1}{2}\|u\|_{H^1}^2$*, and let* $F := F_1 + F_2$*. Then there is a constant* C *independent of* n *such that*

$$\mathbb{E}[F(u(t \wedge \tau_n))] \leq C, \qquad \textit{for all } t \in [0,T]. \tag{8}$$

The proof of Proposition 2.1, which is quite lengthy and technical, is the content of Section 3. Our main result, which relies on Proposition 2.1, is the following.

Theorem 2.2. *Let the assumptions of Proposition 2.1 be satisfied. Then the solution* u *to* (5) *is defined up to time* T *and is* $\mathbb{P}$*-a.s. positive, meaning that*

$$\mathbb{P}\left(u(x,t) > 0 \textit{ for all } x \textit{ in } D \textit{ and for all } t \in [0,T]\right) = 1.$$

Proof. Define $\theta := \frac{\beta}{2} - 1 > 0$. The Hölder inequality and the bound $u^{-\theta} \leq u^{-\beta} + 1$, valid on $(0,\infty)$, give

$$\begin{aligned}
\|u^{-\theta}(t \wedge \tau_n)\|_{W^{1,1}} &= \int_D |u^{-\theta}(t \wedge \tau_n)|\mathrm{d}x + \theta \int_D |u^{-\theta-1}(t \wedge \tau_n)\nabla u(t \wedge \tau_n)|\mathrm{d}x \\
&\leq \int_D |u^{-\theta}(t \wedge \tau_n)|\mathrm{d}x + \theta\left(\int_D |u^{-2(\theta+1)}(t \wedge \tau_n)|\mathrm{d}x\right)^{1/2}\left(\int_D |\nabla u(t \wedge \tau_n)|^2\mathrm{d}x\right)^{1/2} \\
&\leq C + C\int_D |u^{-\beta}(t \wedge \tau_n)|\mathrm{d}x + C\|u(t \wedge \tau_n)\|_{H^1}^2 \leq C + CF(u(t \wedge \tau_n)).
\end{aligned}$$

This immediately entails, using Proposition 2.1, that

$$\mathbb{E}\Big[\|u^{-\theta}(t \wedge \tau_n)\|_{W^{1,1}}\Big] \leq C, \qquad \text{for all } t \in [0,T], \tag{9}$$

where C is independent of n. Let $t \in [0,T]$. We use the $\mathbb{P}$-a.s. H^1-continuity of the paths of u, the continuous embedding $W^{1,1} \hookrightarrow C(0,2\pi)$ (with embedding constant K_1), the Chebyshev inequality, and equations (8) and (9) to deduce

$$\mathbb{P}\left(\tau_n < t\right) \leq \mathbb{P}\left(\min_{x\in D}|u(t\wedge\tau_n)| \leq n^{-1}\right) + \mathbb{P}\left(\|u(t\wedge\tau_n)\|_{H^1} \geq n\right) = \mathbb{P}\left(\max_{x\in D}|u(t\wedge\tau_n)|^{-\theta} \geq n^{\theta}\right)$$
$$+\mathbb{P}\left(\|u(t\wedge\tau_n)\|_{H^1}^2 \geq n^2\right) \leq \mathbb{P}\left(\|u^{-\theta}(t\wedge\tau_n)\|_{W^{1,1}} \geq K_1^{-1}n^{\theta}\right) + \mathbb{P}\left(\|u(t\wedge\tau_n)\|_{H^1}^2 \geq n^2\right)$$
$$\leq \frac{\mathbb{E}[\|u^{-\theta}(t\wedge\tau_n)\|_{W^{1,1}}]}{K_1^{-1}n^{\theta}} + \frac{\mathbb{E}[\|u(t\wedge\tau_n)\|_{H^1}^2]}{n^2} \to 0$$

as $n \to 0$. This implies that $\mathbb{P}\left(\sup_n \tau_n = T\right) = 1$, and concludes the proof. $\square$

3 Proof of Proposition 2.1

We split the proof in four parts. In Subsection 3.1, we compute and properly bound the Itô differential of the process $F(u)$ up to time $t \wedge \tau_n$, for any $t \in [0,T]$. In Subsection 3.2, we group all the terms from the previously computed Itô differential into families, each family being characterised by a specific term. Subsections 3.3 and 3.4 are concerned with imposing conditions on the parameters $p, B_h, p_h, c_h, B_g, p_g, c_g, \gamma_1, \gamma_2$, and $\{\lambda_r\}_{r=0}^{\infty}$ in such a way that (8) is achieved; more specifically, Subsection 3.3 provides the relevant analysis on $A_0 \cup A_\infty$, while Subsection 3.4 consistently extends this analysis on to A_1.

For notational convenience, we rewrite (5) as $\mathrm{d}u = \phi(u(t))\mathrm{d}t + \Phi(u(t))\mathrm{d}\mathcal{W}(t)$, where

$$\phi(u) = \phi_1(u) + \phi_2(u) + \phi_3(u) := -\nabla\cdot\left(m(u)\nabla\left[\Delta u - W'(u)\right]\right) + h(u)|\nabla u|^2 + g(u),$$
$$\Phi(u)v := \nabla\cdot\left(\sqrt{m(u)}v\right).$$

Integration by parts entails that the component of the stochastic noise of (5) along the direction e_i, for $i \in \mathbb{N}_0$, is

$$\left\langle \int_0^t \Phi(u(s))\mathrm{d}\mathcal{W}(s), e_i\right\rangle = \left\langle \int_0^t \nabla\cdot\left(\sqrt{m(u(s))}\sum_{r=0}^{\infty}\sqrt{\lambda_r}e_r\mathrm{d}\beta_r(s)\right), e_i\right\rangle$$
$$= -\left\langle \int_0^t \sqrt{m(u(s))}\sum_{r=0}^{\infty}\sqrt{\lambda_r}e_r\mathrm{d}\beta_r(s), \nabla e_i\right\rangle = \sum_{r=0}^{\infty}\int_0^t -\left\langle\sqrt{m(u(s))}e_r, \nabla e_i\right\rangle\sqrt{\lambda_r}\mathrm{d}\beta_r(s).$$

Thus Φ can be thought of as an infinite-dimensional noise represented with components given by

$$\Phi_{i,r}(u(s)) := -\left\langle\sqrt{m(u(s))}e_r, \nabla e_i\right\rangle, \qquad \text{for all } i, r \in \mathbb{N}_0. \tag{10}$$

3.1 Itô formula for $F(u(t\wedge\tau_n))$

We use the Itô formula

$$G(u(t\wedge\tau_n)) = G(u(0)) + \int_0^{t\wedge\tau_n} G_u(u(s))\phi(u(s))\mathrm{d}s$$
$$+\int_0^{t\wedge\tau_n}\frac{1}{2}\mathrm{Tr}\Big[G_{uu}(u(s))(\Phi(u(s))Q^{\frac{1}{2}})(\Phi(u(s))Q^{\frac{1}{2}})^T\Big]\mathrm{d}s$$
$$+\int_0^{t\wedge\tau_n} G_u(u(s))\Phi(u(s))\mathrm{d}\mathcal{W}(s) + =: I_1 + I_2 + I_3 + I_4, \tag{11}$$

here stated for a real-valued functional G applied to the solution u. We can apply (11) to $G = F_1$ and $G = F_2$ because, up to time $t \wedge \tau_n$: (i) the deterministic and stochastic integrand of (5) satisfy the assumptions of [5, Theorem 4.32]; (ii) F_1 and F_2 are both uniformly continuous (along with their first and second derivatives) over bounded sets of H^1.

We analyse terms I_2, I_3, and I_4 of (11) for $G = F_1$ and $G = F_2$. Time dependence is often dropped for notational convenience.

Term I_2 for $G = F_1$. The first and second derivatives of F_1 are $F_{1,u}(u)v = -\beta \int_D u^{-\beta-1} v \mathrm{d}x$ and $F_{1,uu}(u)(v_1, v_2) = \beta(\beta+1) \int_D u^{-\beta-2} v_1 v_2 \mathrm{d}x$. We study the contributions of ϕ_1, ϕ_2, and ϕ_3 on $F_{1,u}(u)\phi(u)$ separately. We obtain

$$\begin{aligned}
F_{1,u}(u)\phi_1(u) &= \left\langle -\nabla \cdot \left(m(u)\nabla\left[\Delta u - W'(u)\right]\right), -\beta u^{-\beta-1} \right\rangle \\
&= \beta(\beta+1) \left\langle m(u)\nabla[\Delta u - W'(u)], u^{-\beta-2}\nabla u \right\rangle \\
&= \beta(\beta+1) \left\langle \nabla[\Delta u - W'(u)], m(u)u^{-\beta-2}\nabla u \right\rangle = -\beta(\beta+1) \left\langle \Delta u, \nabla(m(u)u^{-\beta-2}\nabla u) \right\rangle \\
&\quad - \beta(\beta+1) \left\langle W''(u)\nabla u, m(u)u^{-\beta-2}\nabla u \right\rangle \\
&= -\beta(\beta+1) \left\langle \Delta u, m(u)u^{-\beta-2}\Delta u \right\rangle - \beta(\beta+1) \left\langle \Delta u, (m(u)u^{-\beta-2})'|\nabla u|^2 \right\rangle \\
&\quad - \beta(\beta+1) \left\langle W''(u)\nabla u, m(u)u^{-\beta-2}\nabla u \right\rangle .
\end{aligned}$$

We remind the reader of the identity

$$\langle f(u)|\nabla u|^2, \Delta u \rangle = -\frac{1}{3} \langle f'(u)|\nabla u|^2, |\nabla u|^2 \rangle , \tag{12}$$

which is valid for $f \in \mathcal{C}^1(0,\infty)$. We choose $f(u) := (m(u)u^{-\beta-2})'$ and deduce

$$\begin{aligned}
F_{1,u}(u)\phi_1(u) &= -\beta(\beta+1) \left\langle \Delta u, m(u)u^{-\beta-2}\Delta u \right\rangle \\
&\quad + \frac{\beta(\beta+1)}{3} \left\langle (m(u)u^{-\beta-2})''|\nabla u|^2, |\nabla u|^2 \right\rangle - \beta(\beta+1) \left\langle W''(u)\nabla u, m(u)u^{-\beta-2}\nabla u \right\rangle .
\end{aligned} \tag{13}$$

As for ϕ_2 and ϕ_3, the contributions are simply

$$F_{1,u}(u)\phi_2(u) = \left\langle h(u)|\nabla u|^2, -\beta u^{-\beta-1} \right\rangle , \qquad F_{1,u}(u)\phi_3(u) = \left\langle g(u), -\beta u^{-\beta-1} \right\rangle . \tag{14}$$

Term I_2 for $G = F_2$. The first and second derivatives of F_2 are $F_{2,u}(u)v = \langle u, v \rangle_{H^1}$ and $F_{2,uu}(u)(v_1, v_2) = \langle v_1, v_2 \rangle_{H^1}$. We study the contributions of ϕ_1, ϕ_2, and ϕ_3 on $F_{2,u}(u)\phi(u)$ separately. We set $f(u) := m(u)W''(u)$ and we obtain, by relying on (12) and using integration by parts

$$\begin{aligned}
&F_{2,u}(u)\phi_1(u) \\
&= \langle -\nabla \cdot (m(u)\nabla\left[\Delta u - W'(u)\right]), u \rangle_{H^1} = \langle -\nabla \cdot (m(u)\nabla\left[\Delta u - W'(u)\right]), u \rangle \\
&\quad + \langle \nabla(-\nabla \cdot (m(u)\nabla\left[\Delta u - W'(u)\right])), \nabla u \rangle \\
&= \langle m(u)\nabla[\Delta u - W'(u)], \nabla u \rangle + \langle \nabla \cdot (m(u)\nabla[\Delta u - W'(u)]), \Delta u \rangle \\
&= \langle \nabla[\Delta u - W'(u)], m(u)\nabla u \rangle - \langle m(u)\nabla[\Delta u - W'(u)], u_{xxx} \rangle \\
&= -\langle \Delta u, m'(u)|\nabla u|^2 \rangle - \langle \Delta u, m(u)\Delta u \rangle - \langle W''(u)\nabla u, m(u)\nabla u \rangle - \langle m(u)u_{xxx}, u_{xxx} \rangle \\
&\quad + \langle f(u)\nabla u, u_{xxx} \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3}\left\langle m''(u)|\nabla u|^2, |\nabla u|^2\right\rangle - \langle \Delta u, m(u)\Delta u\rangle - \langle W''(u)\nabla u, m(u)\nabla u\rangle \\
&\quad - \langle m(u)u_{xxx}, u_{xxx}\rangle - \langle f(u)\Delta u, \Delta u\rangle - \left\langle f'(u)|\nabla u|^2, \Delta u\right\rangle \\
&= \frac{1}{3}\left\langle [m''(u)+f''(u)]\,|\nabla u|^2, |\nabla u|^2\right\rangle - \langle \Delta u, m(u)[1+W''(u)]\Delta u\rangle \\
&\quad - \langle W''(u)\nabla u, m(u)\nabla u\rangle - \langle m(u)u_{xxx}, u_{xxx}\rangle\,. \qquad (15)
\end{aligned}
$$

The contribution associated with ϕ_2 is

$$
\begin{aligned}
F_{2,u}(u)\phi_2(u) &= \left\langle h(u)|\nabla u|^2, u\right\rangle_{H^1} = \left\langle h(u)u, |\nabla u|^2\right\rangle + \left\langle \nabla(h(u)|\nabla u|^2), \nabla u\right\rangle \\
&= \left\langle h(u)u, |\nabla u|^2\right\rangle - \left\langle h(u)|\nabla u|^2, \Delta u\right\rangle = \left\langle h(u)u, |\nabla u|^2\right\rangle + \frac{1}{3}\left\langle h'(u)|\nabla u|^2, |\nabla u|^2\right\rangle, \quad (16)
\end{aligned}
$$

while the contribution associated with ϕ_3 is

$$
F_{2,u}(u)\phi_3(u) = \langle g(u), u\rangle + \langle g'(u)\nabla u, \nabla u\rangle\,. \qquad (17)
$$

Term I_3 for $G = F_1$. We rely on (10) and the expression of $F_{1,uu}$ to compute the Itô correction

$$
\begin{aligned}
&\frac{1}{2}\mathrm{Tr}\left[F_{1,uu}(u)(\Phi(u)Q^{1/2})(\Phi(u)Q^{1/2})^T\right] \\
&\quad = \beta(\beta+1)\sum_{r=0}^{\infty}\lambda_r\sum_{s=0}^{\infty}\sum_{z=0}^{\infty}\left\langle u^{-\beta-2}e_z, e_s\right\rangle\left\langle \sqrt{m(u)}e_r, e_{s,x}\right\rangle\left\langle \sqrt{m(u)}e_r, e_{z,x}\right\rangle. \qquad (18)
\end{aligned}
$$

Remark 3.1. One can convince oneself of the nature of (18) by thinking of a finite-dimensional equivalent of the problem, formulated in terms of the matrices

$$
Q_m = \mathrm{diag}\left\{\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_m}\right\}, \quad [\Phi_m(u)]_{i,r} := -\left\langle \sqrt{m(u)}e_r, \nabla e_i\right\rangle, \quad i,r \in \{0,\cdots,m\}, \qquad (19)
$$

$$
[F_{1,uu}(u)]_m\,(e_i, e_r) = \beta(\beta+1)\int_D u^{-\beta-2}e_i e_r \mathrm{d}x, \quad i,r \in \{0,\cdots,m\}.
$$

We bound (18) by using integration by parts, the Parseval identity in L^2 (for the sums over z and s), the rapid decay of $\{\lambda_r\}_{r=0}^{\infty}$, and the fact that $\|(\mathrm{d}^k/\mathrm{d}x^k)e_r\|_{L^\infty} \le C_k r^k$ (for the sum over r). We obtain

$$
\begin{aligned}
&\beta(\beta+1)\sum_{r=0}^{\infty}\lambda_r\sum_{s=0}^{\infty}\sum_{z=0}^{\infty}\left\langle u^{-\beta-2}e_z, e_s\right\rangle\left\langle \sqrt{m(u)}e_r, e_{s,x}\right\rangle\left\langle \sqrt{m(u)}e_r, e_{z,x}\right\rangle \\
&\quad = \beta(\beta+1)\sum_{r=0}^{\infty}\lambda_r\sum_{s=0}^{\infty}\sum_{z=0}^{\infty}\left\langle u^{-\beta-2}e_z, e_s\right\rangle\left\langle \nabla\left(\sqrt{m(u)}e_r\right), e_s\right\rangle\left\langle \nabla\left(\sqrt{m(u)}e_r\right), e_z\right\rangle \\
&\quad = \beta(\beta+1)\sum_{r=0}^{\infty}\lambda_r\sum_{s=0}^{\infty}\left\langle \nabla\left(\sqrt{m(u)}e_r\right), e_s\right\rangle\left\langle \nabla\left(\sqrt{m(u)}e_r\right), u^{-\beta-2}e_s\right\rangle \\
&\quad = \beta(\beta+1)\sum_{r=0}^{\infty}\lambda_r\left\langle \left|\nabla\left(\sqrt{m(u)}e_r\right)\right|^2, u^{-\beta-2}\right\rangle \qquad (20) \\
&\quad \le C(\beta,\{\lambda\}_r)\left\{\left\langle m^{-1}(u)(m'(u))^2u^{-\beta-2}\,\nabla u, \nabla u\right\rangle + \int_D m(u)u^{-\beta-2}\mathrm{d}x\right\}. \qquad (21)
\end{aligned}
$$

Remark 3.2. Alternatively, one can identify (20) by using [4, Section 3].

Term I_3 for $G = F_2$. We compute the Itô correction

$$\frac{1}{2}\mathrm{Tr}\left[F_{2,uu}(u)(\Phi(u)Q^{1/2})(\Phi(u)Q^{1/2})^T\right] = \sum_{r=0}^{\infty}\lambda_r\sum_{z=0}^{\infty}(1+z^2)\left\langle\sqrt{m(u)}e_r, e_{z,x}\right\rangle^2 \tag{22}$$
$$= \sum_{r=0}^{\infty}\lambda_r\sum_{z=0}^{\infty}\left\langle\sqrt{m(u)}e_r, e_{z,x}\right\rangle^2 + \sum_{r=0}^{\infty}\lambda_r\sum_{z=0}^{\infty}z^2\left\langle\sqrt{m(u)}e_r, e_{z,x}\right\rangle^2 =: T_1 + T_2.$$

Once again, the reader can convince oneself of the nature of (22) by thinking of a finite-dimensional equivalent of the problem, thus relying on the matrices Q_m and $\Phi_m(u)$ defined in (19), as well as on the matrix $[F_{2,uu}(u)]_m = \mathrm{diag}\{(1+z^2)\}_{z=1,\cdots,m}$. See Remark 3.1 also.
We bound T_2. Given the nature of the trigonometric basis $\{e_r\}_{r=0}^{\infty}$, we have (for $r \geq 1$), that $re_{r,x} = \delta(r)\Delta e_{\sigma(r)}$, for some injective function $\sigma : \mathbb{N} \to \mathbb{N}$ and where $\delta(r) \in \{-1; +1\}$. We use integration by parts and the Parseval identity (for the sum over z) and obtain

$$T_2 = \sum_{r=0}^{\infty}\lambda_r\sum_{z=0}^{\infty}\left\langle\sqrt{m(u)}e_r, \Delta e_z\right\rangle^2 = \sum_{r=0}^{\infty}\lambda_r\sum_{z=0}^{\infty}\left\langle\Delta\left(\sqrt{m(u)}e_r\right), e_z\right\rangle^2 = \sum_{r=0}^{\infty}\lambda_r\left\|\Delta\left(\sqrt{m(u)}e_r\right)\right\|^2 \tag{23}$$
$$\leq C\sum_{r=0}^{\infty}\lambda_r\left[\left\|\left\{-\frac{1}{4}m^{-3/2}(m')^2 + \frac{1}{2}m^{-1/2}(u)m''(u)\right\}|\nabla u|^2 e_r\right\|^2 + \left\|\frac{1}{2}m^{-1/2}(u)m'(u)\Delta u e_r\right\|^2\right. \tag{24}$$
$$\left. + \left\|m^{-1/2}(u)m'(u)\nabla u\, e_{r,x}\right\|^2 + \left\|\sqrt{m(u)}\Delta e_r\right\|^2\right]$$
$$\leq C(\{\lambda_r\}_r)\left\{\langle\left[m^{-1}(u)(m''(u))^2 + m^{-3}(u)(m'(u))^4\right]|\nabla u|^2, |\nabla u|^2\rangle\right.$$
$$\left. + \langle m^{-1}(u)(m'(u))^2\Delta u, \Delta u\rangle + \langle m^{-1}(u)(m'(u))^2\nabla u, \nabla u\rangle + \int_D m(u)\mathrm{d}x\right\}, \tag{25}$$

where the right-hand-side of (23) can also be inferred from [4, Section 3].

Remark 3.3. Given the polynomial nature of $m(u)\,|_{A_0\cup A_\infty}$, it is easy to notice that the multiplying term $T_3 := -\frac{1}{4}m^{-3/2}(m')^2 + \frac{1}{2}m^{-1/2}(u)m''(u)$ in (24) vanishes if and only if $\gamma_1 = \gamma_2 = 2$. In all other cases, the terms making up T_3 are proportional to each other.

As for T_1, the computation is simpler, and it reads, thanks to the Parseval inequality

$$T_1 = \sum_{r=0}^{\infty}\lambda_r\sum_{z=0}^{\infty}\left\langle\sqrt{m(u)}e_r, e_{z,x}\right\rangle^2 = \sum_{r=0}^{\infty}\lambda_r\sum_{z=0}^{\infty}\left\langle\nabla\left(\sqrt{m(u)}e_r\right), e_z\right\rangle^2 = \sum_{r=0}^{\infty}\lambda_r\left\|\nabla\left(\sqrt{m(u)}e_r\right)\right\|^2 \tag{26}$$
$$\leq C\sum_{r=0}^{\infty}\lambda_r\left[\left\|m^{-1/2}(u)m'(u)\nabla u\, e_r\right\|^2 + \left\|\sqrt{m(u)}e_{r,x}\right\|^2\right] \tag{27}$$
$$\leq C(\{\lambda_r\}_r)\left\{\langle m^{-1}(u)(m'(u))^2\nabla u, \nabla u\rangle + \int_D m(u)\mathrm{d}x\right\},$$

where the right-hand-side of (26) can once again be inferred from [4, Section 3]. We deduce

$$\frac{1}{2}\mathrm{Tr}\left[F_{2,uu}(u)(\Phi(u)Q^{1/2})(\Phi(u)Q^{1/2})^T\right]$$
$$\leq C(\{\lambda_r\}_r)\left\{\langle\left[m^{-1}(u)(m''(u))^2 + m^{-3}(u)(m'(u))^4\right]|\nabla u|^2, |\nabla u|^2\rangle\right.$$
$$\left. + \langle m^{-1}(u)(m'(u))^2\Delta u, \Delta u\rangle + \langle m^{-1}(u)(m'(u))^2\nabla u, \nabla u\rangle + \int_D m(u)\mathrm{d}x\right\}, \tag{28}$$

Term I_4 for $G = F_1$. We rely on [5, Theorem 4.36] and bound the Itô isometry term associated with I_4. We use integration by parts and the Parseval identity to deduce

$$\begin{aligned}
&\sum_{r=0}^{\infty} \lambda_r \left| \sum_{z=0}^{\infty} -\beta \left\langle u^{-\beta-1}, e_z \right\rangle \left\langle \nabla \left(\sqrt{m(u)} e_r \right), e_z \right\rangle \right|^2 = \beta \sum_{r=0}^{\infty} \lambda_r \left| \left\langle u^{-\beta-1}, \nabla \left(\sqrt{m(u)} e_r \right) \right\rangle \right|^2 \\
&\quad = \beta \sum_{r=0}^{\infty} \lambda_r \left| \left\langle (\beta+1) u^{-\beta-2} \nabla u, \sqrt{m(u)} e_r \right\rangle \right|^2 \leq C(\{\lambda_r\}_r, \beta) \left\langle u^{-\beta-2} m(u) \nabla u, u^{-\beta-2} \nabla u \right\rangle . \qquad (29)
\end{aligned}$$

Given the definition of τ_n, we deduce that I_4 is a square-integrable martingale with mean zero, see [5, Proposition 4.28]. The contribution of I_4 can thus be neglected.

Term I_4 for $G = F_2$. Again relying on [5, Theorem 4.36], we bound the Itô isometry term associated with I_2. Similarly to (29), we deduce

$$\begin{aligned}
&\sum_{r=0}^{\infty} \lambda_r \left| \sum_{z=0}^{\infty} (1+z^2) \left\langle u, e_z \right\rangle \left\langle \nabla \left(\sqrt{m(u)} e_r \right), e_z \right\rangle \right|^2 = \sum_{r=0}^{\infty} \lambda_r \left| \sum_{z=0}^{\infty} \left\langle u, e_z - \Delta e_z \right\rangle \left\langle \nabla \left(\sqrt{m(u)} e_r \right), e_z \right\rangle \right|^2 \\
&\quad \leq \sum_{r=0}^{\infty} 2\lambda_r \left| \left\langle u, \nabla \left(\sqrt{m(u)} e_r \right) \right\rangle \right|^2 + \sum_{r=0}^{\infty} 2\lambda_r \left| \left\langle \Delta u, \nabla \left(\sqrt{m(u)} e_r \right) \right\rangle \right|^2 \\
&\quad = \sum_{r=0}^{\infty} 2\lambda_r \left| \left\langle \nabla u, \sqrt{m(u)} e_r \right\rangle \right|^2 + \sum_{r=0}^{\infty} 2\lambda_r \left| \left\langle u_{xxx}, \sqrt{m(u)} e_r \right\rangle \right|^2 \\
&\quad \leq C(\{\lambda_r\}_r) \left\{ \left\langle \nabla u, m(u) \nabla u \right\rangle + \left\langle u_{xxx}, m(u) u_{xxx} \right\rangle \right\} . \qquad (30)
\end{aligned}$$

In this case, the definition of τ_n does not imply that I_4 is a square-integrable martingale with mean zero. This is due to the presence of the term $\langle u_{xxx}, m(u) u_{xxx} \rangle$.

3.2 Clustering contributions from the Itô formula

In the previous section we have provided bounds for the terms I_2, I_3, I_4 associated with the Itô formula applied to the functionals $F_1(u)$ and $F_2(u)$. These bounds contain terms which can be clustered in five distinct families, identified as

$$\int_D p(u), \qquad \text{(F1)}$$

$$\langle p(u) \Delta u, \Delta u \rangle , \qquad \text{(F2)}$$

$$\langle p(u) |\nabla u|^2, |\nabla u|^2 \rangle , \qquad \text{(F3)}$$

$$\langle p(u) \nabla u, \nabla u \rangle , \qquad \text{(F4)}$$

$$\langle p(u) u_{xxx}, u_{xxx} \rangle , \qquad \text{(F5)}$$

for some $p \in \mathcal{C}(0, \infty)$. Notice that all contributions to the Itô formula are well defined, because of assumption (6). With the exception of the terms in the right-hand-side of (29) (associated with the Itô isometry of I_4 for the functional $F_1(u)$), we now cluster all the terms belonging to the same family.

Terms of kind (F1). Relevant terms are gathered from (28), (21), (17), (14), adding up to

$$C(\{\lambda_r\}_r) \int_D m(u) \mathrm{d}x + C(\{\lambda_r\}_r, \beta) \int_D m(u) u^{-\beta-2} \mathrm{d}x + \langle g(u), u \rangle + \left\langle g(u), -\beta u^{-\beta-1} \right\rangle . \qquad (31)$$

Terms of kind (F2). Relevant terms are gathered from (13), (15), (28), adding up to

$$\begin{aligned}&-\beta(\beta+1)\left\langle \Delta u, m(u)u^{-\beta-2}\Delta u\right\rangle - \langle \Delta u, m(u)\Delta u\rangle - \langle \Delta u, m(u)W''(u)\Delta u\rangle \\ &+ C(\{\lambda_r\}_r)\left\langle m^{-1}(u)(m'(u))^2\Delta u, \Delta u\right\rangle . \end{aligned} \tag{32}$$

Terms of kind (F3). Relevant terms are gathered from (13), (15), (16), (28), adding up to

$$\begin{aligned}&C(\beta)\left\langle (m(u)u^{-\beta-2})''|\nabla u|^2, |\nabla u|^2\right\rangle + \frac{1}{3}\left\langle m''(u)|\nabla u|^2, |\nabla u|^2\right\rangle + \frac{1}{3}\left\langle (m(u)W''(u))''|\nabla u|^2, |\nabla u|^2\right\rangle \\ &+ \frac{1}{3}\left\langle h'(u)|\nabla u|^2, |\nabla u|^2\right\rangle + C(\{\lambda_r\}_r)\left\langle \left[m^{-1}(u)(m''(u))^2 + m^{-3}(u)(m'(u))^4\right]|\nabla u|^2, |\nabla u|^2\right\rangle . \end{aligned} \tag{33}$$

Terms of kind (F4). Relevant terms are gathered from (13), (14), (15), (16), (21), (28), (17), (30), adding up to

$$\begin{aligned}&-\beta(\beta+1)\left\langle W''(u)\nabla u, m(u)u^{-\beta-2}\nabla u\right\rangle - C(\beta)\left\langle h(u)|\nabla u|^2, u^{-\beta-1}\right\rangle - \langle W''(u)\nabla u, m(u)\nabla u\rangle \\ &+ \left\langle h(u)u, |\nabla u|^2\right\rangle + C(\{\lambda\}_r)\left\langle m^{-1}(u)(m'(u))^2u^{-\beta-2}\nabla u, \nabla u\right\rangle \\ &+ C(\{\lambda\}_r)\left\langle m^{-1}(u)(m'(u))^2\nabla u, \nabla u\right\rangle \\ &+ \langle g'(u)\nabla u, \nabla u\rangle + C(\{\lambda\}_r)\langle \nabla u, m(u)\nabla u\rangle . \end{aligned} \tag{34}$$

Terms of kind (F5). Relevant terms are gathered from (15), (30), adding up to

$$(C(\{\lambda_r\}_r) - 1)\langle m(u)u_{xxx}, u_{xxx}\rangle . \tag{35}$$

3.3 Parameter tuning on $A_0 \cup A_\infty$

We now look for conditions on the parameters $p, B_h, p_h, c_h, B_g, p_g, c_g, \gamma_1, \gamma_2$, and $\{\lambda_r\}_{r=0}^\infty$ in order to obtain (8). More specifically, we look for conditions on these parameters in such a way that some of the terms in (31), (32), (33), (34), and (35) can be bounded by the two Gronwall type terms $\int_D u^{-\beta}$ and $\|u\|_{H^1}^2$, while the remaining can be bounded by constants. In order to easily identify the relevant parameters, for each of the families (F1)–(F4) we draw two summary tables. As for the first table:

(i) each column is associated with a term of the family in question, the terms being listed in order of appearance in the corresponding expression among (31), (32), (33), and (34).

(ii) the second row shows the degree of the monomial restriction $p(u)\,|_{A_0}$.

(iii) the first row shows the constants multiplying $p(u)\,|_{A_0}$.

We will denote this kind of table by $\mathcal{A}_0$. As for the second table, everything is defined in the same way, but with the region A_0 replaced by A_∞. We will denote this kind of table by $\mathcal{A}_\infty$. We deal with the analysis on the region A_1 in the following subsection.

Summary table and conditions for family (F1). Tables $\mathcal{A}_0$ and $\mathcal{A}_\infty$ summarising (31) are given in Figure 2. Condition (C3) insures that the leading polynomial order is contained in the fourth (respectively, third) column for $\mathcal{A}_0$ (respectively, $\mathcal{A}_\infty$). The contribution given by the family (F1) is then bounded by a constant.

Summary table and conditions for family (F2). Tables $\mathcal{A}_0$ and $\mathcal{A}_\infty$ summarising (32) are given in Figure 3. For both $\mathcal{A}_0$ and $\mathcal{A}_\infty$, the only positive contribution comes from column 4. This

$$\mathcal{A}_0 = \begin{array}{|c|c|c|c|}\hline C(\{\lambda_r\}) & C(\beta,\{\lambda_r\}) & B_g & -\beta B_g \\ \hline \gamma_1 & \gamma_1-\beta-2 & -p_g+1 & -\beta-1-p_g \\ \hline \end{array}$$

$$\mathcal{A}_\infty = \begin{array}{|c|c|c|c|}\hline C(\{\lambda_r\}) & C(\beta,\{\lambda_r\}) & -B_g & \beta B_g \\ \hline \gamma_2 & \gamma_2-\beta-2 & c_g+1 & -\beta-1+c_g \\ \hline \end{array}$$

Figure 2: Tables $\mathcal{A}_0$ and $\mathcal{A}_\infty$ for family (F1).

$$\mathcal{A}_0 = \begin{array}{|c|c|c|c|}\hline -C(\beta) & -1 & -p(p+1) & \gamma_1^2 C(\{\lambda_r\}_r) \\ \hline \gamma_1-\beta-2 & \gamma_1 & \gamma_1-p-2 & \gamma_1-2 \\ \hline \end{array}$$

$$\mathcal{A}_\infty = \begin{array}{|c|c|c|c|}\hline -C(\beta) & -1 & -p(p+1) & \gamma_2^2 C(\{\lambda_r\}_r) \\ \hline \gamma_2-\beta-2 & \gamma_2 & \gamma_2-p-2 & \gamma_2-2 \\ \hline \end{array}$$

Figure 3: Tables $\mathcal{A}_0$ and $\mathcal{A}_\infty$ for family (F2).

contribution can be compensated, e.g., with column 1 (in the case of $\mathcal{A}_0$) or column 2 (in the case of $\mathcal{A}_\infty$) by using (C1).

Summary table and conditions for family (F3). Tables $\mathcal{A}_0$ and $\mathcal{A}_\infty$ summarising (33) are given in Figure 4. For $\mathcal{A}_0$ (respectively, $\mathcal{A}_\infty$) we can pick p_h, B_h big enough (respectively, c_h, B_h big enough)

$$\mathcal{A}_0 = \begin{array}{|c|c|c|c|c|}\hline C(\gamma_1,\beta) & C(\gamma_1) & p(p+1)(\gamma_1-p-2)(\gamma_1-p-3) & -p_h B_h/3 & C(\gamma_1)C(\{\lambda_r\}_r) \\ \hline \gamma_1-\beta-4 & \gamma_1-2 & \gamma_1-p-4 & -p_h-1 & \gamma_1-4 \\ \hline \end{array}$$

$$\mathcal{A}_\infty = \begin{array}{|c|c|c|c|c|}\hline C(\gamma_2,\beta) & C(\gamma_2) & p(p+1)(\gamma_2-p-2)(\gamma_2-p-3) & -c_h B_h/3 & C(\gamma_2)C(\{\lambda_r\}_r) \\ \hline \gamma_2-\beta-4 & \gamma_2-2 & \gamma_2-p-4 & c_h-1 & \gamma_2-4 \\ \hline \end{array}$$

Figure 4: Tables $\mathcal{A}_0$ and $\mathcal{A}_\infty$ for family (F3).

so that column 4 contains the leading-order monomial, with also sufficiently big multiplicative constant. Thus column 4 compensates all the other columns. We have thus invoked (C2).

Summary table and conditions for family (F4). Tables $\mathcal{A}_0$ and $\mathcal{A}_\infty$ summarising (34) are given in Figure 5.
If we invoke (C3) for both $\mathcal{A}_0$ and $\mathcal{A}_\infty$, then column 7 contains the leading order. Thus all other columns are compensated by a constant.

Conditions for family (F5). Contribution (35) is negative as long as we invoke (C1).

3.4 Parameter tuning on A_1

Conditions (C1)-(C3) are also enough to provide the same conclusions, as in Subsection 3.3, for the families (F1)-(F5) analysed over A_1. More specifically: the domain D being bounded, the continuity of m does not alter the estimate for the family (F1); the estimate for the family (F2) still holds due to (C1) and (4a); the estimate for the family (F3) still holds due to (4a)–(4b) and (C2); the estimate for the family (F4) still holds, due to (4a) and the fact that we are allowed to bound everything with a constant times $|\nabla u|^2$, so there is no issue in the local behaviour in a neighbourhood of $u = 1$.

$\mathcal{A}_0 =$

$-C(\beta)p(p+1)$	$-C(\beta)B_h$	$-p(p+1)$	B_h	$C(\{\lambda_r\}_r,\gamma_1)$	$C(\{\lambda_r\}_r,\gamma_1)$	$-B_g p_g$	1
$\gamma_1-\beta-p-4$	$-p_h-\beta-1$	γ_1-p-2	$-p_h+1$	$\gamma_1-\beta-4$	γ_1-2	$-p_g-1$	γ_1

$\mathcal{A}_\infty =$

$-C(\beta)p(p+1)$	$C(\beta)B_h$	$-p(p+1)$	$-B_h$	$C(\{\lambda_r\}_r,\gamma_2)$	$C(\{\lambda_r\}_r,\gamma_2)$	$-B_g c_g$	1
$\gamma_2-\beta-p-4$	$c_h-\beta-1$	γ_2-p-2	c_h+1	$\gamma_2-\beta-4$	γ_2-2	c_g-1	γ_2

Figure 5: Tables $\mathcal{A}_0$ and $\mathcal{A}_\infty$ for family (F4).

Finally, thanks to (4a), nothing needs to be adapted for the family (F5).

We can complete the proof of Proposition 2.1 by taking the expected value in the Itô formula (11) for $G = F_1 + F_2$.

4 Analysis of results

We compare our setting to that of J. Fischer and F. Grün, whose paper [7] has inspired us to this work. In [7], existence of a $\mathbb{P}$-a.s. positive solution to the conservative thin-film equation (i.e., equation (2) with $h \equiv g \equiv 0$) is established in the case of quadratic mobility $m(u) = u^2$. This specific mobility, corresponding to $\gamma_1 = \gamma_2 = 2$ in our notation, results in a linear stochastic noise which makes h and g unnecessary in the argument. We detail this last statement by making a direct comparison to our computations.

No need for h. No term belonging to the family (F3) arises when $\gamma_1 = \gamma_2 = 2$. Firstly, the Itô correction applied to $\|\nabla u\|^2$ does not produce any such term, because of the linear nature of $\sqrt{m(u)} = u$, see Remark 3.3. We can thus drop the (F3)-term in (28), which corresponds to column 5 in $\mathcal{A}_0$ and $\mathcal{A}_\infty$. Secondly, if one picks $p := \beta > 2$ (this is compatible with the setting in [7]), some computations can be performed better. In particular, one can combine the drift contributions coming from the Itô formula applied to functional $F_3(u) := \|\nabla u\|^2 + F_1(u)$, thus getting, for $p_r := -\Delta u + W'(u)$

$$
\begin{aligned}
&\langle u_x, \nabla(-\nabla(m(u)\nabla(-p_r)))\rangle + \langle W'(u), -\nabla(m(u)\nabla(-p_r))\rangle \\
&\quad= \langle \Delta u, \nabla(m(u)\nabla(-p_r))\rangle + \langle \nabla[W'(u)], m(u)\nabla(-p_r)\rangle \\
&\quad= -\langle \nabla[\Delta u], m(u)\nabla(-p_r)\rangle + \langle \nabla[W'(u)], m(u)\nabla(-p_r)\rangle = -\langle p_{r,x}, m(u)p_{r,x}\rangle \le 0.
\end{aligned}
$$

The above computation is a way of regrouping relevant drift terms in a slightly differently way. More specifically, the final term $\langle p_{r,x}, m(u)p_{r,x}\rangle$ can be rewritten as

$$
\langle m(u)u_{xxx}, u_{xxx}\rangle + \langle W''(u)\nabla u, m(u)W''(u)\nabla u\rangle - 2\langle u_{xxx}, m(u)W''(u)\nabla u\rangle
$$

and the last term in above expression contains the contributions of columns 1 and 3 of $\mathcal{A}_0$ and $\mathcal{A}_\infty$ (which coincide, as $\beta = p$, see (13) and (15)). Finally, column 2 of $\mathcal{A}_0$ and $\mathcal{A}_\infty$ is dealt with by not computing the Itô formula for $\|u\|^2$ at all, as one relies on Poincaré inequality arguments based on the conservation of mass. One is then left only with column 4 of $\mathcal{A}_0$ and $\mathcal{A}_\infty$, which are associated with h.

Remark 4.1. In [7], the quantity $-\langle p_{r,x}, m(u)p_{r,x}\rangle$ is used to balance the Itô isometry term coming from the stochastic noise given by a suitable combination of F_1 and F_2. In this paper, we have analysed F_1 and F_2 separately, thus the quantity $-\langle p_{r,x}, m(u)p_{r,x}\rangle$ has not quite emerged.

No need for g. This follows under the weaker assumptions $\gamma_2 \le 2$, $2 \le \gamma_1 \le 2+\beta$. The first term in (31) is of Gronwall type, simply because

$$\int_D m(u)\mathrm{d}x \le C + \|u\|_{L^{\gamma_2}}^{\gamma_2} \le C + C\|u\|_{H^1}^2.$$

As for the second term in (31), it is also of Gronwall type. We write

$$C(\{\lambda_r\}_r)\int_D m(u)u^{-\beta-2}\mathrm{d}x \le C(\{\lambda_r\}_r)\int_D u^{\gamma_1-\beta-2}\mathrm{d}x + C(\{\lambda_r\}_r)\int_D u^{\gamma_2-\beta-2}\mathbf{1}_{u>1+\epsilon}\mathrm{d}x + C.$$

This yields

$$C(\{\lambda_r\}_r)\int_D m(u)u^{-\beta-2}\mathrm{d}x \le C(\{\lambda_r\}_r)\int_D u^{\gamma_1-\beta-2}\mathrm{d}x + C.$$

For $2 \le \gamma_1 < \beta+2$ and $\beta > 2$ we get that $-\beta/(\gamma_1-\beta-2) \ge 1$. We use the Hölder inequality to obtain

$$C(\{\lambda_r\}_r)\int_D u^{\gamma_1-\beta-2}\mathrm{d}x \le C(\{\lambda_r\}_r)\left(\int_D u^{-\beta}\mathrm{d}x\right)^{\frac{\gamma_1-\beta-2}{-\beta}} \le C(\{\lambda_r\}_r)\int_D u^{-\beta}\mathrm{d}x + C.$$

When $\gamma_1 = \beta+2$, the above inequality is also trivially valid. This means that columns 1 and 2 of $\mathcal{A}_0$ and $\mathcal{A}_\infty$ for the family (F1) are bounded by Gronwall terms, and g is thus superfluous.

Remark 4.2. It is worth noticing that, in the conservative case with quadratic mobility, the potential W is actually needed. The potential W is only involved in bounding all the terms in family (F4), while it is not necessary to deal with the families (F1), (F2), (F3), and (F5). In the non-conservative case with mobility $m(u)$ not being quadratic, the use of W can be bypassed by properly tuning h, which is needed for the family (F3) anyway. As a matter of fact, we can not use W only, and we may actually not use it at all, as h carries the leading order.

The contents of this section have shown that the potential h is concerned with addressing nonlinearities of the stochastic noise of (2) (i.e., analysis for $\gamma_1 \neq 2$ or $\gamma_2 \neq 2$), while g is concerned with being able to deal with noise of "large" size in regimes of both low and high density u (i.e., analysis for $\gamma_1 < 2$ and $\gamma_2 > 2$). In particular, the terms $h(u)|\nabla u|^2$ and $g(u)$ appear to be a plausible drift correction for the specific case of the Dean-Kawasaki model in (1), which corresponds to $\gamma_1 = \gamma_2 = 1$.

5 Considerations on a Galerkin discretisation of the problem

In this work we have dealt with an *a priori* regularity analysis for solutions to (2). More specifically, we have assumed the existence of a local regular solution to (2), and we have shown that it can be extended up to any given time $T > 0$ while also being positive $\mathbb{P}$-a.s. We devote this section to explaining the major difficulties one encounters when trying to prove existence of local solutions to (2) in the conservative case (corresponding to $h \equiv 0$, $g \equiv 0$).

One may rely on a Galerkin scheme for a spatial discretisation of the problem. Two natural basis choices come up: (i) the trigonometric basis, see Subsection 1.1; (ii) the hat basis for the space of periodic linear finite elements on the uniform grid $\{0, h, 2h, \cdots, 2\pi - h, 2\pi\}$, where h in an integer fraction of 2π, see [7].

The use of the trigonometric basis might seem slightly more suitable to deal with the differential operators of (2). However, it is subject to a disadvantage. For $m := 2\pi h^{-1}$, let u_m be the solution to the m-dimensional Galerkin approximation of (2) with respect to an L^2-projection onto $V_m := \{e_1, \cdots, e_m\}$. It is not hard to see that computing the Itô formula for the functional $F(u_m)$, where F is the same as in Proposition 2.1, leads to a few terms carrying a projection operator π_m onto V_m. In particular, one gets such a projection for the drift component associated with F_1. This is an issue, as having projections on the drift term annihilates the compensation that such term could potentially have on the positive terms coming from the Itô correction for F_1 and F_2. One can avoid the appearance of such projections by only considering quadratic quantities in u_m, such as $F_2(u_m)$. However, one loses any indication of positivity of the solutions u_m, which may only be defined up to a random time τ; this is primarily due to the function W not being bounded near the origin, thus preventing us from using the standard existence theory (see, e.g., [9, Chapter IV, Theorem 2.2]). One can not get around this issue by simply smoothening the potential W near the origin, as to do so would not provide uniform estimates for $\mathbb{E}[F(u_m)]$; one can intuitively see this from the summary tables given in Subsection 3.3.

On the other hand, the use of the hat basis proved to be successful in [7] in the case of quadratic mobility. We limit ourselves to briefly summarising the two main reasons for this. Firstly, one may rely on the so called *entropy consistency* for the discrete mobility [8], which allows to discretise the quadratic mobility in a piecewise constant function, for the benefit of relevant integral equations and of projection purposes onto the finite-dimensional Galerkin approximation space. Secondly, the solution u_m being piecewise linear, it has piecewise constant derivative $u_{m,x}$. This fact allows to detach contributions involving the quadratic term $|u_{m,x}|^2$ from the contribution given by the (nonlinear) term $W''(u_m)$, thus simplifying the analysis. Moreover, the contribution given by $W''(u_m)$ is in turn provided by the hat basis spatial discretisation of the problem, which allows to suitably bound the ratios of the values of u_m at adjacent grid nodes. These key observations allow the authors in [7] to effectively deal with the nonlinearities of the problem, represented by the quadratic mobility and polynomial potential W, within the framework of a Galerkin scheme associated with both positivity and appropriate tightness arguments for the solutions u_m. However, this Galerkin approximation scheme does not seem to be extendable (at least in the conservative case) to mobilities whose square roots have unbounded first derivatives, i.e., in which either $\gamma_1 < 2$ or $\gamma_2 > 2$. One can find a justification of the previous statement by keeping in mind our discussion for the need of h and g given in Section 4.

6 Conclusions

For equation (2), non-conservative contributions h and g appear to be necessary in order to show a priori positivity of solutions in the case of non-quadratic mobility m. The role of h is to compensate for nonlinearities arising from the Itô calculus associated with relevant functionals of the unknown process u, while the role of g is to compensate for large noise in low and high density regimes. In particular, the Dean-Kawasaki model seems to require a drift correction. The a priori positivity analysis has been performed by using a functional representation with respect to the trigonometric basis of L^2. Establishing existence of local solutions (which could then be extended up to any time $T > 0$ while preserving positivity) seems to be unpractical if one is to use a Galerkin approximation scheme with respect to this basis; in the conservative case, there seems to be a good chance to prove existence of positive solutions with a Galerkin scheme with respect to the hat basis, but only in the case of mobilities whose square roots have bounded first derivatives ($\gamma_1 > 2$ and $\gamma_2 < 2$). If one is to consider different ranges of γ_1 and γ_2, then non-conservative corrections could be of use within the hat basis discretisation framework.

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4.2. Conclusions

We have considered a non-conservative version of the stochastic thin-film equation endowed with specific source potentials, namely, (2). Subject to the existence of a sufficiently regular solution defined up to a random time, we have provided conditions on the source potentials and the stochastic noise in order to extend such solution up to any deterministic finite time (Theorem 2.2). Such result is based on a priori estimates which are uniform w.r.t. the minimum and maximum (positive) values attained by the solution. These estimates (proved in Proposition 2.1) follow from an application of the Itô formula to the relevant functional F, which encodes the solution positivity requirement by controlling the quantity $\|u^{-\theta}\|_{W^{1,1}}$, for some $\theta > 0$. The proof of Proposition 2.1 identifies conditions on the source potentials and the noise in such a way that all families of terms (F1)-(F5) originated by the application of the Itô formula are suitably controlled.

Despite not providing an existence proof for the solution to this thin-film equation, this work gives some insight on the structure of the DK class. Firstly, we give an indication of what the 'magnitude' of the source potentials should be (w.r.t. the mobility coefficient/noise) in order for the model to guarantee (a priori) positivity of solutions. Secondly, we are able to make a parallel to the reference work [24], by showing consistency of the two settings in the case of quadratic mobility. More specifically, in Section 4, we argue that the source potentials are not necessary in the case of quadratic mobility (and this is indeed the case in [24]), and we associate the appearance of some distinctive terms among (F1)-(F5) to a non-quadratic mobility.

We conclude by arguing (Sections 5 and 6), in accordance to the conclusions of [24], that there appears to be substantial difficulties in proving existence of positive solutions to a thin-film equations featuring either a non-Lipschitz or a superlinear noise term $\sqrt{m(\rho)}$.

Chapter 5

Conclusions

In this thesis, we discussed various modelling and analytical aspects for a class of stochastic equations (which we have referred to as the Dean–Kawasaki class) by focusing our attention on two specific incarnations: the DK equation, and a thin-film equation.

In Chapter 2, we enquired whether, and to which extent, regularising the original DK equation (proposed in [15, 32]) can diminish its ill-posedness. We proposed a regularised model in the case of independent particles. In contrast to [15], we considered particle of finite rather than atomic size, where the particle size is related to their total number. This entailed several useful consequences. Firstly, we were able to provide rigorous and precise estimates as for the 'difference' between the microscopic and mesoscopic representation of the noise in the model. Secondly, we obtained suitable tightness for the model in the limit of the number of particles. Thirdly, we showed existence and uniqueness of smooth solutions, in a high-probability sense, for the resulting wave equation with mesoscopic DK type noise: key features here were the small-noise regime of the noise, as well as its smooth Gaussian driving term.

In Chapter 3 we extended the analysis to the case of particles interacting through a pairwise potential, thus generalising our model to more interesting and relevant systems. We relied on a suitable propagation of chaos result to refer the analysis of the weakly interacting particle system in question to an auxiliary system of independent McKean–Vlasov type particles. We quantified the 'price' that one pays to switch between the two systems, and concluded that it is negligible (as the number of particles goes to infinity) due to the chosen regularisation. Using this fact, we were then able to refer to the contents of Chapter 2 in many points, with the notable exception of the tightness argument, which we adapted as a result of the decreased time regularity of the model entailed by the propagation of chaos. We solved the resulting DK model analogously to Chapter 2, with the addition of localisation techniques required to deal with the superlinear interaction term $\{W' * \rho_\epsilon\}\rho_\epsilon$.

Finally, in Chapter 4, we picked up an open question from Chapters 2 and 3 (i.e., can we guarantee almost sure positivity of solutions to our regularised DK models?) and discussed it for a wider set of members of the DK class. More specifically, we took inspiration from [24], and considered non-conservative modifications of a stochastic thin-film equation. We showed that, in the presence of a sufficiently regular local positive solution, such solution can be extended (while remaining positive) up to any finite time, provided that the non-conservative source potentials give a strong enough 'repulsive singularity' (w.r.t. to relevant features of the stochastic noise) at the null density profile. We then put our result into context by highlighting analogies with [24], and drew some conclusions w.r.t. the DK class.

5.1. Possible future directions

Several problems concerning the DK class remain, to this date, open. Among the most important ones, is the issue of positivity of solutions. This natural question applies primarily to both Chapters 2 and 3. There, we have performed a low temperature approximation (e.g., see the approximation of $j_{2,\epsilon}$ in [12, Approximation 3]) in order to close our regularised DK models. This approximation, which is applicable only in local equilibrium, leads to a noise-perturbed wave equation. This equation's drift does not contain singularities which can 'push the solution away' from the null profile. It would be interesting to see whether a more sophisticated approximation of $j_{2,\epsilon}$, possibly in an out-of-equilibrium regime, could provide such repulsive singularities.

On a related note, the DK class is formulated on the level of densities which depend on position and time, but not on velocity. There is thus a significant gap between the DK class and the Vlasov–Fokker–Planck type equations (see for instance [19]), where densities also depend on velocity. In fact, the hydrodynamic limit which is used to derive Vlasov–Fokker–Planck type equations allows for an explicit identification of the equations' drifts. As our regularised DK models are reduced models with half the space dimensions, it is unsurprising that more difficulties arise when trying to close them in the relevant densities ρ_ϵ, j_ϵ. It would be interesting to investigate whether one can 'bridge' between the Vlasov–Fokker-Planck class and the DK class in order to get a better understanding of the deterministic drift for the latter (in the specific case of our regularised DK models, a better understanding of $j_{2,\epsilon}$). On this matter, introducing particle mass in the underlying Langevin dynamics and investigating vanishing mass limits seems to be a relevant starting point [51].

The matter of repulsive singularities in the drift is naturally associated with the contents of Chapter 4. With respect to the contents of this chapter, it would interesting if one could prove the existence of local positive solutions, instead of taking this as an assumption. More specifically, it would be interesting to see whether the numerical schemes found in [24, 27] can be adapted in the case of general monomial mobility and repulsive source potentials.

Additionally, it would be good to gain additional insight in the scaling $N\epsilon^\theta = 1$, which we have used in Chapters 2 and 3, and whose heuristic meaning has been given in Subsection 1.8. From an analytical perspective, it would be interesting to investigate how much θ can be lowered without altering the current results (we have already encountered sub-optimal bounds in our analysis). From a physics perspective, it would be interesting to see whether there is a specific choice of θ which leads to a meaningful representation of $j_{2,\epsilon}$, or to a non-trivial noise in the macroscopic limit (even though this goes beyond the small-noise regime analysis we have so far carried out). From a numerical perspective, we have never conducted numerical simulations for relevant smoothed densities. It would be interesting to understand whether, and in which sense, the scaling affects the computational efficiency for the simulation of the densities $(\rho_\epsilon, j_\epsilon)$. This question applies in the case where $(\rho_\epsilon, j_\epsilon)$ is associated with the 'exact' Langevin particle system, and also in the case where it is a solution to the regularised DK models we have derived in Chapters 2 and 3.

Finally, it would be a significant achievement to lift the DK class analysis to higher spacial dimensions, even though this appears to be challenging.

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